

Long time existence of regular solutions to non-homogeneous Navier-Stokes equations

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Abstract. We consider the motion of incompressible viscous non-homogeneous fluid described by the Navier-Stokes equations in a bounded cylinder Ω under boundary slip conditions. Assume that the x_3 -axis is the axis of the cylinder. Let ϱ be the density of the fluid, v – the velocity and f the external force field. Assuming that quantities $\nabla\varrho(0)$, $\partial_{x_3}v(0)$, $\partial_{x_3}f$, $f_3|_{\partial\Omega}$ are sufficiently small in some norms we prove large time regular solutions such that $v \in H^{2+s,1+s/2}(\Omega \times (0, T))$, $\nabla p \in H^{s,s/2}(\Omega \times (0, T))$, $\frac{1}{2} < s < 1$ without any restriction on the existence time T . The proof is divided into two parts. First an a priori estimate is shown. Next the existence follows from the Leray-Schauder fixed point theorem.

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1. Introduction

We consider the motion of a viscous non-homogeneous incompressible fluid described by the following system

$$\begin{aligned}
 (1.1) \quad & \varrho(v_{,t} + v \cdot \nabla v) - \operatorname{div} \mathbb{T}(v, p) = \varrho f && \text{in } \Omega^T = \Omega \times (0, T), \\
 & \operatorname{div} v = 0 && \text{in } \Omega^T, \\
 & \varrho_{,t} + v \cdot \nabla \varrho = 0 && \text{in } \Omega^T, \\
 & v \cdot \bar{n} = 0 && \text{on } S^T = S \times (0, T), \\
 & \bar{n} \cdot \mathbb{T}(v, p) \cdot \bar{\tau}_\alpha + \delta_{i1} \gamma v \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_i^T, \quad i = 1, 2, \\
 & v|_{t=0} = v_0 && \text{in } \Omega, \\
 & \varrho|_{t=0} = \varrho_0 && \text{in } \Omega,
 \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded cylindrical domain, $S = \partial\Omega$ is the boundary of Ω , $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the fluid, $\varrho = \varrho(x, t) \in \mathbb{R}_+$ the density, $p = p(x, t) \in \mathbb{R}$ the pressure, $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ the external force field and $x = (x_1, x_2, x_3)$ are the Cartesian coordinates. Moreover, \bar{n} is the unit outward vector normal to S , $\bar{\tau}_i$, $i = 1, 2$, are tangent vectors to S and the dot denotes the scalar product in \mathbb{R}^3 . Finally $\gamma \geq 0$ is the constant slip coefficient and δ_{ij} is the Kronecker δ .

By $\mathbb{T}(v, p)$ we denote the stress tensor of the form

$$(1.2) \quad \mathbb{T}(v, p) = \nu \mathbb{D}(v) - p \mathbb{I},$$

where ν is the constant viscosity coefficient, \mathbb{I} is the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$(1.3) \quad \mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

Ω is cylindrical domain parallel to the x_3 -axis with arbitrary cross section. We assume that $S = S_1 \cup S_2$, where S_1 is the part of the boundary parallel to the x_3 -axis and S_2 is perpendicular to x_3 . Hence,

$$\begin{aligned}
 (1.4) \quad & S_1 = \{x \in \mathbb{R}^3 : \psi(x_1, x_2) = c_0, -a < x_3 < a\}, \\
 & S_2 = \{x \in \mathbb{R}^3 : \psi(x_1, x_2) < c_0, x_3 \in \{-a, a\}\},
 \end{aligned}$$

where a and c_0 are given positive numbers and $\psi(x_1, x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$.

Now we formulate the main result of this paper

Theorem A. Assume that

1. $v_0 \in H^{1+s}(\Omega)$, $v_{0,x_3} \in H^1(\Omega)$, $s \in (1/2, 1)$;
2. $\varrho_0 \in W_q^2(\Omega)$, $3 < q \leq \frac{3}{3/2-s}$, $s \in (1/2, 1)$;
3. there exist positive constants $0 < \varrho_* < \varrho^*$ such that $\varrho_* \leq \varrho_0 \leq \varrho^*$;
4. $f \in H^{s,s/2}(\Omega^T)$, $f_{,x_3} \in L_2(\Omega^T)$, $s \in (1/2, 1)$;
5. the quantity

$$(1.4) \quad \begin{aligned} X(T) = & \|\varrho_{0,x}\|_{W_q^1(\Omega)} + \|f_{,x_3}\|_{L_2(0,T;L_{6/5}(\Omega))} \\ & + \|v_{0,x_3}\|_{L_2(\Omega)} + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))} \leq \delta \end{aligned}$$

where δ is sufficiently small.

Then there exists a unique solution to problem (1.1) such that $v \in H^{2+s,1+s/2}(\Omega^T)$, $v_{,x_3} \in H^{2,1}(\Omega^T)$, $\nabla p \in H^{s,s/2}(\Omega^T)$, $\nabla p_{,x_3} \in L_2(\Omega^T)$

$$(1.5) \quad \begin{aligned} & \|v\|_{H^{2+s,1+s/2}(\Omega^T)} + \|v_{,x_3}\|_{H^{2,1}(\Omega^T)} + \|\nabla p\|_{H^{s,s/2}(\Omega^T)} \\ & + \|\nabla p_{,x_3}\|_{L_2(\Omega^T)} \leq \varphi(\varrho_*, \varrho^*, N) \end{aligned}$$

where φ is an increasing positive function and

$$\begin{aligned} N = & \|f\|_{H^{s,s/2}(\Omega^T)} + \|f_{,x_3}\|_{L_2(\Omega^T)} + \|f\|_{L_2(0,T;W_{6/5}^1(\Omega))} \\ & + \|v_0\|_{H^{1+s}(\Omega)} + \|v_{0,x_3}\|_{H^1(\Omega)}. \end{aligned}$$

The result formulated in Theorem A describes a long time existence of solutions to problem (1.1) because the smallness condition (1.4) contains at most time integral norms of f .

The aim of this paper is to prove long time existence of regular solutions to problem (1.1) such that there is no restriction on the magnitudes of the external force, the initial velocity and the density. The aim is covered by the smallness restriction (1.4) because it contains derivatives of the initial density and derivatives with respect to x_3 of the initial velocity and the external force. This kind of restrictions suggests that our solution remains close to two-dimensional solutions of incompressible Navier-Stokes equations because the initial density is close to a constant but the initial velocity and the external force change a little in the x_3 -direction. In view of the result on long time existence of solutions to two-dimensional incompressible nonhomogeneous Navier-Stokes equations (see [AKM, Ch. 3]) we could expect that smallness of $\varrho_{0,x}$ can be replaced by smallness of ϱ_{0,x_3} only.

However, up to now, we do not know how to do it.

One could expect that looking for solutions close to two-dimensional solutions is nothing to do comparing with [AKM, Ch. 3]. But it is totally not true because we need three-dimensional imbeddings, solvability of three-dimensional problems and apply the three-dimensional technique of Sobolev and Sobolev-Slobodetski spaces.

Moreover, we have to mention that many techniques used in this paper were developed in [Z2, Z3, Z4, RZ] in the case of a constant density.

The next step in our considerations will be a global existence result which can be proved by extending [Z4, NZ1] to the nonhomogeneous fluids.

Finally we expect an existence of global attractor by applying the technique of [NZ2].

Many results on existence and estimates of weak solutions to nonhomogeneous incompressible Navier-Stokes equations can be found in [P].

2. Notation

We use isotropic and anisotropic Lebesgue spaces $L_p(Q)$, $Q \in \{\Omega^T, S^T, \Omega, S\}$, $p \in [1, \infty]$; $L_q(0, T; L_p(Q))$, $Q \in \{\Omega, S\}$, $p, q \in [1, \infty]$; isotropic and anisotropic Sobolev spaces with the norms

$$\|u\|_{W_p^s(Q)} = \left(\sum_{|\alpha| \leq s} \int_Q |D_x^\alpha u|^p dx \right)^{1/p}$$

and

$$\|u\|_{W_p^{s,s/2}(Q^T)} = \left(\sum_{|\alpha|+2a \leq s} \int_{Q^T} |D_x^\alpha \partial_t^a u|^p dx dt \right)^{1/p}, \quad s \text{ even},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $a, \alpha_i \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0$, $i = 1, 2, 3$, $Q \in \{\Omega, S\}$, $s \in \mathbb{N}$, $p \in [1, \infty]$.

In the case $p = 2$ we use the notation

$$H^s(Q) = W_2^s(Q), \quad H^{s,s/2}(Q^T) = W_2^{s,s/2}(Q^T), \quad Q \in \{\Omega, S\}.$$

Moreover, $L_2(Q) = H^0(Q)$, $L_p(Q) = W_p^0(Q)$, $L_p(Q^T) = W_p^{0,0}(Q^T)$.

Next we introduce a space natural for examining weak solutions to the Navier-Stokes and parabolic equations

$$\begin{aligned} \|u\|_{V_2^k(\Omega^T)} = \left\{ u : \|u\|_{V_2^k(\Omega^T)} = \operatorname{ess\,sup}_{t \in [0, T]} \|u(t)\|_{H^k(\Omega)} \right. \\ \left. + \left(\int_0^T \|\nabla u\|_{H^k(\Omega)}^2 dt \right)^{1/2} < \infty \right\}, \quad k \in \mathbb{N}_0. \end{aligned}$$

In the case of noneven s spaces $W_p^s(Q)$ and $W_p^{s,s/2}(Q^T)$ are defined as sets of functions with the finite norms, respectively,

$$\|u\|_{W_p^s(Q)} = \left(\sum_{|\alpha| \leq [s]} \int_Q |D_x^\alpha u(x)|^p dx + \sum_{|\alpha| = [s]} \int_Q \int_Q \frac{|D_x^\alpha u(x) - D_{x'}^\alpha u(x')|^p}{|x - x'|^{n+p(s-[s])}} dx dx' \right)^{1/p},$$

and

$$\begin{aligned} \|u\|_{W_p^{s,s/2}(Q^T)} &= \left(\sum_{|\alpha|+2a \leq [s]} \int_{Q^T} |D_x^\alpha \partial_t^a u|^p dx dt + \sum_{|\alpha|+2a=[s]} \int_0^T \int_Q \int_Q \frac{|D_x^\alpha \partial_t^a u(x,t) - D_{x'}^\alpha \partial_t^a u(x',t)|^p}{|x - x'|^{n+p(s-[s])}} dx dx' dt \right. \\ &\quad \left. + \sum_{|\alpha|+2a=[s]} \int_Q \int_0^T \int_0^T \frac{|D_x^\alpha \partial_t^a u(x,t) - D_{x'}^\alpha \partial_{t'}^a u(x,t')|^p}{|t - t'|^{1+p(s/2-[s/2])}} dx dt dt' \right)^{1/p}, \end{aligned}$$

where $Q \subset \mathbb{R}^n$, $[s]$ the integer parts of s .

In the case where either $s = [s]$ or $\frac{s}{2} = [\frac{s}{2}]$ the corresponding fractional derivatives vanish. For $Q = S$ the above norms are defined by applying a partition of unity.

Theorems of imbedding and interpolation for above spaces can be found in [BIN].

By $\dot{C}^\alpha(\Omega^T)$ we denote a space of functions with the finite seminorm

$$\|u\|_{\dot{C}^\alpha(\Omega^T)} = \sup_{x, x' \in \Omega} \sup_{t, t' \in (0, T)} \frac{|u(x, t) - u(x', t')|}{|x - x'|^\alpha + |t - t'|^\alpha},$$

where $\alpha \in (0, 1)$.

By c we denote a generic constant which changes its value from formula to formula. In general c depends on constants of imbeddings, on functions describing the boundary, but it does not depend on data. By φ we denote a generic function which depends on data, changes its form from formula to formula and is always positive increasing function of its arguments.

The dependence of φ on data will be always expressed explicitly.

To simplify presentation we use the notation

$$\varrho_x^* = \|\varrho_x\|_{L_\infty(\Omega^T)}.$$

Let us consider the Stokes system

$$\begin{aligned}
(2.1) \quad & v_t - \operatorname{div} \mathbb{T}(v, p) = f' && \text{in } \Omega^T, \\
& \operatorname{div} v = 0 && \text{in } \Omega^T, \\
& v \cdot \bar{n} = 0 && \text{on } S^T, \\
& \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \delta_{i1} \gamma v \cdot \bar{\tau}_\alpha = h_\alpha, \quad \alpha = 1, 2, && \text{on } S_i^T, \quad i = 1, 2, \\
& v|_{t=0} = v_0 && \text{in } \Omega.
\end{aligned}$$

From [Z3] we have

Lemma 2.1. *Assume that $f' \in W_2^{s,s/2}(\Omega^T)$, $h_\alpha \in W_2^{s+1/2,s/2+1/4}(S^T)$, $\alpha = 1, 2$, $s \in \mathbb{R}_+ \cup \{0\}$, $v_0 \in W_2^{s+1}(\Omega)$, $S \in C^{[s]+3}$, where $[s]$ is the integer part of s . Then there exists a unique solution to problem (2.1) such that $v \in W_2^{s+2,s/2+1}(\Omega^T)$, $\nabla p \in W_2^{s,s/2}(\Omega^T)$ and*

$$\begin{aligned}
(2.2) \quad & \|v\|_{W_2^{s+2,s/2+1}(\Omega^T)} + \|\nabla p\|_{W_2^{s,s/2}(\Omega^T)} \\
& \leq c \left(\|f'\|_{W_2^{s,s/2}(\Omega^T)} + \|v_0\|_{W_2^{s+1}(\Omega)} + \sum_{\alpha=1}^2 \|h_\alpha\|_{W_2^{s+1/2,s/2+1/4}(S^T)} \right).
\end{aligned}$$

After small modifications of the proof from [A1] we have

Lemma 2.2. *Assume that $f' \in L_r(\Omega^T)$, $h_\alpha \in W_r^{1-1/r,1/2-1/2r}(S^T)$, $r \in (1, \infty)$, $S \in C^2$, $v_0 \in W_r^{2-2/r}(\Omega)$. Then there exists a solution to problem (2.1) such that $v \in W_r^{2,1}(\Omega^T)$, $\nabla p \in L_r(\Omega^T)$ and*

$$\begin{aligned}
(2.3) \quad & \|v\|_{W_r^{2,1}(\Omega^T)} + \|\nabla p\|_{L_r(\Omega^T)} \leq c \left(\|f'\|_{L_r(\Omega^T)} \right. \\
& \left. + \sum_{\alpha=1}^2 \|h_\alpha\|_{W_r^{1-1/r,1/2-1/2r}(S^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)} \right).
\end{aligned}$$

Let us consider the problem

$$\begin{aligned}
(2.4) \quad & \varrho v_t - \operatorname{div} \mathbb{T}(v, p) = \varrho f' && \text{in } \Omega^T, \\
& \operatorname{div} v = 0 && \text{in } \Omega^T, \\
& v \cdot \bar{n} = 0 && \text{on } S^T, \\
& \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \delta_{i1} \gamma v \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S^T, \quad i = 1, 2, \\
& v|_{t=0} = v_0 && \text{in } \Omega.
\end{aligned}$$

Using a partition of unity, next the result from [A1] in the half-space and finally applying a perturbation argument we have

Lemma 2.3. *Let the assumptions of Lemma 2.2 hold. Let $\varrho \in \dot{C}^\alpha(\Omega^T)$, $\alpha \in (0, 1)$, $\varrho, 1/\varrho \in L_\infty(\Omega^T)$, $\nabla \varrho \in L_\infty(\Omega^T)$ and there exist positive constants $0 < \varrho_* < \varrho^*$ such that $\varrho_* \leq \varrho \leq \varrho^*$. Let $v \in L_r(\Omega^T)$, $p \in L_r(\Omega^T)$. Then for solutions to problem (2.4) the following inequality holds*

$$(2.5) \quad \begin{aligned} & \|v\|_{W_r^{2,1}(\Omega^T)} + \|\nabla p\|_{L_r(\Omega^T)} \leq \varphi(\varrho_*, \varrho^*, \|\nabla \varrho\|_{L_\infty(\Omega^T)}) [\|v\|_{L_r(\Omega^T)} \\ & + \|p\|_{L_r(\Omega^T)} + \|f'\|_{L_r(\Omega^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)}]. \end{aligned}$$

By a partition of unity and the result from [Z3] in the half-space a perturbation argument implies

Lemma 2.4. *Let the assumptions of Lemma 2.1 be satisfied. Let $\varrho, \frac{1}{\varrho}, \varrho_x, \varrho_t \in L_\infty(\Omega^T)$ and let there exist positive constants $0 < \varrho_* < \varrho^*$ such that $\varrho_* \leq \varrho \leq \varrho^*$. Let $s \in \mathbb{R}_+ \cup \{0\}$ and let $\varrho \in W_\infty^{s,s/2}(\Omega^T)$. Let $v \in L_2(\Omega^T)$ and $p \in W_2^{s,s/2}(\Omega^T)$. Then there exists a solution to problem (2.4) such that $v \in W_2^{s+2,s/2+1}(\Omega^T)$, $\nabla p \in W_2^{s,s/2}(\Omega^T)$, $s \in (0, 1)$ and*

$$(2.6) \quad \begin{aligned} & \|v\|_{W_2^{2+s,1+s/2}(\Omega^T)} + \|\nabla p\|_{W_2^{s,s/2}(\Omega^T)} \\ & \leq \varphi(\varrho_*, \varrho^*) (1 + \|\partial_x \varrho\|_{L_\infty(\Omega^T)} + \|\partial_t \varrho\|_{L_\infty(\Omega^T)}) \\ & \quad \cdot [\|v\|_{L_2(\Omega^T)} + \|p\|_{H^{s,s/2}(\Omega^T)} + \|f'\|_{H^{s,s/2}(\Omega^T)} + \|v_0\|_{H^{1+s}(\Omega)}], \end{aligned}$$

where φ is the generic function.

3. Auxiliary results

This section is devoted to obtain some a priori estimates for solutions to problem (1.1). Therefore, we assume existence of such solutions to (1.1) that the derived estimates can be satisfied.

First we introduce weak solutions

Definition 3.1. By a weak solution to problem (1.1) we mean $v \in V_2^0(\Omega^T)$, $\varrho \in L_\infty(\Omega^T)$ such that $\operatorname{div} v = 0$, $v \cdot \bar{n}|_S = 0$ and satisfying the integral identities

$$(3.1) \quad \begin{aligned} & \int_{\Omega^T} [-\varrho v \phi_t - \varrho v \otimes v \cdot \nabla \phi + \frac{\nu}{2} \mathbb{D}(v) \cdot \mathbb{D}(\phi)] dx dt \\ & + \gamma \int_{S_1} v \cdot \bar{\tau}_\alpha \phi \cdot \bar{\tau}_\alpha dS_1 - \int_{\Omega} \varrho_0 v_0 \phi|_{t=0} dx = \int_{\Omega^T} \varrho f \phi dx dt, \\ & \int_{\Omega^T} [\varrho \psi_{,t} + \varrho v \cdot \nabla \psi] dx dt + \int_{\Omega} \varrho_0 \psi|_{t=0} dx = 0, \end{aligned}$$

for any $\phi, \psi \in W_{5/2}^{1,1}(\Omega^T)$ such that $\phi \cdot \bar{n}|_S = 0$, $\operatorname{div} \phi = 0$, $\phi(T) = 0$, $\psi(T) = 0$ and the summation convention over the repeated indices is assumed.

We need the Korn inequality

Lemma 3.2. (see [Z1]) Assume that

$$(3.2) \quad E_\Omega(v) = \|\mathbb{D}(v)\|_{L_2(\Omega)}^2 < \infty, \quad v \cdot \bar{n}|_S = 0, \quad \operatorname{div} v = 0.$$

If Ω is not axially symmetric there exists a constant c_1 such that

$$(3.3) \quad \|v\|_{H^1(\Omega)}^2 \leq c_1 E_\Omega(v).$$

If Ω is axially symmetric, $\eta = (-x_2, x_1, 0)$, $\alpha = \int_\Omega v \cdot \eta dx$ is bounded then there exists a constant c_2 such that

$$(3.4) \quad \|v\|_{H^1(\Omega)}^2 \leq c_2 \left(E_\Omega(v) + \left| \int_\Omega v \cdot \eta dx \right|^2 \right).$$

Let us consider the problem

$$(3.5) \quad \begin{aligned} \varrho_t + v \cdot \nabla \varrho &= 0 & \text{in } \Omega^T, \\ \varrho|_{t=0} &= \varrho_0 & \text{in } \Omega. \end{aligned}$$

Lemma 3.3. Let $\varrho_0 \in L_p(\Omega)$, $p \in \mathbb{R} \setminus \{0\}$, $\operatorname{div} v = 0$, $v \cdot \bar{n}|_S = 0$. Assume that there exists a sufficiently regular solution to problem (3.5). Then the a priori equality holds

$$(3.6) \quad \|\varrho\|_{L_\infty(0,T;L_p(\Omega))} = \|\varrho_0\|_{L_p(\Omega)}.$$

Proof. Let $p \leq 1$ and $\varrho \in C^1(\Omega^T)$. Multiplying (3.5) by $\varrho|\varrho|^{p-2}$ and integrating over Ω^T yields, by the density argument, the equality

$$\|\varrho\|_{C(0,T;L_p(\Omega))} = \|\varrho_0\|_{L_p(\Omega)}.$$

Hence (3.6) holds.

For $p < 1$, $p \neq 0$ we assume additionally that ϱ, ϱ_0 are different from zero. Hence, performing the same considerations as in the case $p \geq 1$, we obtain the same equality as above. Finally, (3.6) also holds for $p < 1$, $p \neq 0$. This concludes the proof.

Remark 3.4. We can pass with p to $+\infty$ and $-\infty$ in (3.6). Let ϱ_* , ϱ^* be positive constants such that

$$(3.7) \quad \varrho_* \leq \varrho_0 \leq \varrho^*.$$

Then (3.6) implies

$$(3.8) \quad \varrho_* \leq \varrho(x, t) \leq \varrho^*.$$

Next we formulate a result about weak solutions

Lemma 3.5. Assume that Ω is not axially symmetric. Assume that $f \in L_1(0, T; L_2(\Omega))$, $v_0 \in L_2(\Omega)$, $\varrho_0, \frac{1}{\varrho_0} \in L_\infty(\Omega)$. Assume that there exist constants ϱ_* , ϱ^* described in Remark 3.4 and (3.7) holds. Then a weak solution to problem (1.1) is such that $v \in V_2^0(\Omega^T)$ and $\varrho_* \leq \varrho(x, t) \leq \varrho^*$. Moreover, we have the a priori estimates

$$(3.9) \quad \|v(t)\|_{L_2(\Omega)} \leq \left(\frac{\varrho^*}{\varrho_*}\right)^{1/2} d_1 = d_2, \quad t \leq T,$$

$$(3.10) \quad \|v\|_{V_2^0(\Omega^t)} = \left(\frac{\varrho^*}{c_3}\right)^{1/2} d_1 \equiv d_3, \quad t \leq T,$$

where $c_3 = \min\left(\frac{1}{2}\varrho_*, \nu c_1\right)$ and

$$(3.11) \quad d_1(T) = \|f\|_{L_1(0, T; L_2(\Omega))} + \|v_0\|_{L_2(\Omega)}.$$

Proof. Assume that we have the existence of sufficiently regular solutions to (1.1). Multiplying (1.1)₁ by v , integrating over Ω and using (1.1)_{2,3,4} yields

$$(3.12) \quad \frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho}v\|_{L_2(\Omega)}^2 + \gamma \|\mathbb{D}(v)\|_{L_2(\Omega)}^2 + \gamma \int_{S_1} |v \cdot \bar{\tau}_\alpha|^2 dS_1 = \int_{\Omega} \varrho f \cdot v dx.$$

Omitting the second and the third terms on the l.h.s. and applying the Hölder inequality to the r.h.s. implies

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho}v\|_{L_2(\Omega)}^2 \leq \|\sqrt{\varrho}f\|_{L_2(\Omega)} \|\sqrt{\varrho}v\|_{L_2(\Omega)}.$$

Hence we get

$$(3.13) \quad \frac{d}{dt} \|\sqrt{\varrho}v\|_{L_2(\Omega)} \leq \|\sqrt{\varrho}f\|_{L_2(\Omega)}.$$

Integrating (3.13) with respect to time yields

$$(3.14) \quad \|\sqrt{\varrho}v(t)\|_{L_2(\Omega)} \leq \|\sqrt{\varrho}f\|_{L_1(0, T; L_2(\Omega))} + \|\sqrt{\varrho_0}v_0\|_{L_2(\Omega)},$$

where $t \leq T$.

Using (3.7) and (3.8) in (3.14) gives (3.9).

Integrating (3.12) with respect to time and using (3.3) we obtain

$$(3.15) \quad \begin{aligned} & \frac{1}{2} \|\sqrt{\varrho} v\|_{L_2(\Omega)}^2 + \nu c_1 \int_0^t \|v\|_{H^1(\Omega)}^2 dt' \leq \|\sqrt{\varrho} f\|_{L_1(0,t;L_2(\Omega))} \\ & \cdot \operatorname{ess\,sup}_{t' \leq t} \|\sqrt{\varrho} v\|_{L_2(\Omega)} + \frac{1}{2} \|\sqrt{\varrho_0} v_0\|_{L_2(\Omega)}^2. \end{aligned}$$

In view of (3.14), (3.7) and (3.8) we have (3.10). This concludes the proof. To prove the existence with large data we follow the ideas developed in [RZ, Z2,Z4]. therefore we introduce the quantities

$$(3.16) \quad h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3}, \quad \chi = (\operatorname{rot} v)_3, \quad F = (\operatorname{rot} f)_3.$$

Lemma 3.6. *Let v, ϱ be given. Then (h, q) is a solution to the problem*

$$(3.17) \quad \begin{aligned} & \varrho h_{,t} - \operatorname{div} \mathbb{T}(h, q) = -\varrho(v \cdot \nabla h + h \cdot \nabla v - g) \\ & \quad - \varrho_{,x_3}(v_{,t} + v \cdot \nabla v - f) && \text{in } \Omega^T, \\ & \operatorname{div} h = 0 && \text{in } \Omega^T, \\ & h \cdot \bar{n} = 0, \quad \bar{n} \cdot \mathbb{T}(h, q) \cdot \bar{\tau}_\alpha + \gamma h \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\ & h_i = 0, \quad i = 1, 2, \quad h_{3,x_3} = 0 && \text{on } S_2^T, \\ & q|_{S_2} = \varrho f_3 && \text{on } S_2^T, \\ & h|_{t=0} = h(0) && \text{in } \Omega. \end{aligned}$$

Proof. (3.17)_{1,2,3,6} follow directly from (1.1)_{1,2,4,5,6} by differentiation with respect to x_3 . Similarly as in [Z2] we show the boundary condition (3.17)₄. This ends the proof.

To formulate problem for χ we introduce

$$(3.18) \quad \begin{aligned} & \bar{n}|_{S_1} = \frac{\nabla \psi}{|\nabla \psi|} = \frac{1}{|\nabla \psi|}(\psi_{,x_1} \psi_{,x_2}, 0), \\ & \bar{\tau}_1|_{S_1} = \frac{\nabla^\perp \psi}{|\nabla \psi|} = \frac{1}{|\nabla \psi|}(-\psi_{,x_2}, \psi_{,x_1}, 0), \\ & \bar{\tau}_2|_{S_1} = (0, 0, 1), \\ & \bar{n}|_{S_2} = (0, 0, 1), \quad \bar{\tau}_1|_{S_2} = (1, 0, 0), \quad \bar{\tau}_2|_{S_2} = (0, 1, 0). \end{aligned}$$

Lemma 3.7. *Let ϱ, v, h be given. Then χ is a solution to the problem (3.19)*

$$\begin{aligned}
& \varrho(\chi_{,t} + v \cdot \nabla \chi) - \nu \Delta \chi = \varrho(F + \chi h_3 - v_{3,x_1} h_2 + v_{3,x_2} h_1) \\
& \quad + \varrho_{,x_1}(v_{2,t} + v \cdot \nabla v_2 + f_2) - \varrho_{,x_2}(v_{1,t} + v \cdot \nabla v_1 + f_1) \quad \text{in } \Omega^T, \\
& \chi = v_i(n_{i,x_j} \tau_{1j} + \tau_{1i,x_j} n_j) + v \cdot \bar{\tau}_1(\tau_{12,x_1} - \tau_{11,x_2}) \\
& \quad + \frac{\gamma}{\nu} v_j \tau_{1j} \equiv \chi_* \quad \text{on } S_1^T, \\
& \chi_{,x_3} = 0 \quad \text{on } S_2^T, \\
& \chi|_{t=0} = \chi(0) \quad \text{in } \Omega,
\end{aligned}$$

where the summation convention over the repeated indices is assumed.

Proof. (3.19)₁ follows from applying two-dimensional rot operator to the first two equations of (1.1)₁. The boundary condition (3.19)₂ was proved in [Z2]. This ends the proof.

To apply the energy type method to problem (3.19) we need

Lemma 3.8. *Let χ satisfy (3.19). Let $\tilde{\chi}$ be a solution to the problem*

$$\begin{aligned}
& \varrho \tilde{\chi}_{,t} - \nu \Delta \tilde{\chi} = 0 \quad \text{in } \Omega^T, \\
& \tilde{\chi} = \chi_* \quad \text{on } S_1^T, \\
& \tilde{\chi}_{,x_3} = 0 \quad \text{on } S_2^T, \\
& \tilde{\chi}|_{t=0} = 0 \quad \text{in } \Omega.
\end{aligned}
\tag{3.20}$$

Then the function $\chi' = \chi - \tilde{\chi}$ satisfies

$$\begin{aligned}
& \varrho(\chi'_t + v \cdot \nabla \chi') - \nu \Delta \chi' \\
& \quad = \varrho(F + \chi h_3 - v_{3,x_1} h_2 + v_{3,x_2} h_1 - v \cdot \nabla \tilde{\chi}) \\
& \quad \quad + \varrho_{,x_1}(f_2 + v_{2,t} + v \cdot \nabla v_2) \\
& \quad \quad - \varrho_{,x_2}(f_1 + v_{1,t} + v \cdot \nabla v_1) \quad \text{in } \Omega^T, \\
& \chi'|_{S_1} = 0, \\
& \chi'_{,x_3}|_{S_2} = 0, \\
& \chi'|_{t=0} = \chi(0).
\end{aligned}
\tag{3.21}$$

Let us consider problem (3.5). Then we have

Lemma 3.9. *Assume that $v \in W_2^{s,s/2}(\Omega^T)$, $s > \frac{5}{2}$. Let*

$$X_1 = \|\varrho_x(0)\|_{L_\infty(\Omega)} + \|\varrho_t(0)\|_{L_\infty(\Omega)}.
\tag{3.22}$$

Then the following a priori estimate is valid

$$(3.23) \quad \begin{aligned} & \|\varrho_x(t)\|_{L_p(\Omega)} + \|\varrho_t(t)\|_{L_p(\Omega)} + \|\varrho\|_{\dot{C}^\alpha(\Omega^T)} \leq \varphi(\|v\|_{W_2^{s,s/2}(\Omega^T)})X_1, \\ & \alpha \leq 1 - 1/p, \quad p \leq \infty, \end{aligned}$$

where

$$(3.24) \quad \|\varrho\|_{\dot{C}^\alpha(\Omega^T)} = \sup_{\substack{x, x' \in \Omega \\ t, t' \in (0, T)}} \frac{|\varrho(x, t) - \varrho(x', t')|}{|x - x'|^\alpha + |t - t'|^\alpha},$$

where $|x - x'| = \sum_{i=1}^3 |x_i - x'_i|$.

Proof. For solutions to problem (3.5) we have

$$\frac{d}{dt} \|\varrho_x\|_{L_p(\Omega)} \leq \|v_x\|_{L_\infty(\Omega)} \|\varrho_x\|_{L_p(\Omega)},$$

for any $p > 1$.

Integrating with respect to time yields

$$(3.25) \quad \|\varrho_x(t)\|_{L_p(\Omega)} \leq \exp\left(\int_0^t \|v_x(t')\|_{L_\infty(\Omega)} dt'\right) \|\varrho_x(0)\|_{L_p(\Omega)},$$

where $p \in [1, \infty]$.

From (3.5)₁ we obtain

$$(3.26) \quad \|\varrho_t(t)\|_{L_p(\Omega)} \leq \|v\|_{L_\infty(\Omega^T)} \|\varrho_x\|_{L_p(\Omega)}.$$

Let us consider the expression

$$(3.27) \quad \begin{aligned} & \sup_{\substack{x, x' \in \Omega \\ t, t' \in [0, T]}} \frac{|\varrho(x, t) - \varrho(x', t')|}{\sum_{i=1}^3 |x_i - x'_i|^{1-1/p} + |t - t'|^{1-1/p}} \\ & \leq \sup_{\substack{x, x' \in \Omega \\ t, t' \in [0, T]}} \left[\frac{|\varrho(x_1, x_2, x_3, t) - \varrho(x'_1, x_2, x_3, t)|}{|x_1 - x'_1|^{1-1/p}} \right. \\ & \quad + \frac{|\varrho(x'_1, x_2, x_3, t) - \varrho(x'_1, x'_2, x_3, t)|}{|x_2 - x'_2|^{1-1/p}} \\ & \quad \left. + \frac{|\varrho(x'_1, x'_2, x_3, t) - \varrho(x'_1, x'_2, x'_3, t)|}{|x_3 - x'_3|^{1-1/p}} + \frac{|\varrho(x', t) - \varrho(x', t')|}{|t - t'|^{1-1/p}} \right] \\ & \leq \sup_{\substack{x, x' \in \Omega \\ t, t' \in [0, T]}} \left[\sum_{i=1}^3 \left(\int_{x'_i}^{x_i} |\varrho_{,x_i}|^p dx_i \right)^{1/p} + \left(\int_{t'}^t |\varrho_{,t}|^p dt \right)^{1/p} \right] \\ & \leq c(\|\varrho_x\|_{L_\infty(\Omega^T)} + \|\varrho_t\|_{L_\infty(\Omega^T)}). \end{aligned}$$

Using the imbedding

$$(3.28) \quad \|v\|_{L_\infty(\Omega^T)} + \|v_x\|_{L_2(0,T;L_\infty(\Omega))} \leq c\|v\|_{W_2^{s,s/2}(\Omega^T)}$$

for $s > \frac{5}{2}$, we obtain from (3.25)–(3.27) estimate (3.23). This concludes the proof.

Lemma 3.10. *Let ϱ and v be given and sufficiently regular. Assume also that vectors \bar{n} , $\bar{\tau}_\alpha$, $\alpha = 1, 2$, are defined in a neighbourhood of S and $a_{\alpha\beta}$, a depend on $D_x^\sigma \bar{n}$, $D_x^\sigma \bar{\tau}_\alpha$, $\alpha = 1, 2$, $\sigma \leq 2$. Then p is a solution to the problem*

$$(3.29) \quad \begin{aligned} \Delta p &= -\nabla \varrho \cdot v_t - \operatorname{div}(\varrho v \cdot \nabla v) + \operatorname{div}(\varrho f) \quad \text{in } \Omega, \\ \frac{\partial p}{\partial n} \Big|_{S_j} &= (\varrho v_i v_j n_{j,x_i} + \nu a_{j\alpha\beta} v_{\tau_\alpha, \tau_\beta} + \nu a_j \cdot v + \varrho f \cdot \bar{n})|_{S_j}, \quad j = 1, 2, \\ \int_\Omega p dx &= 0, \end{aligned}$$

where the last equation was added to have uniqueness of solutions to (3.29).

Proof. Applying div to (1.1)₁ we get (3.29)₁. Multiplying (1.1)₁ by \bar{n} and projecting on S we obtain the boundary condition

$$(3.30) \quad \begin{aligned} \frac{\partial p}{\partial n} \Big|_S &= (-\varrho \bar{n} \cdot v_t - \varrho n_i v \cdot \nabla v_i + \nu \bar{n} \cdot \Delta v + \varrho f \cdot \bar{n})|_S \\ &= (\varrho v_i v_j n_{j,x_i} + \nu \bar{n} \cdot \Delta v + \varrho f \cdot \bar{n})|_S \equiv I, \end{aligned}$$

where we used that $v \cdot \bar{n}|_S = 0$ and the summation convention over the repeated indices is assumed.

Now we calculate $\bar{n} \cdot \Delta v|_S$. Let us introduce the curvilinear coordinates n , τ_α , $\alpha = 1, 2$, connected with the orthonormal system of vectors \bar{n} , $\bar{\tau}_\alpha$, $\alpha = 1, 2$. Then we calculate

$$(3.31) \quad \begin{aligned} \Delta v &= v_{,x_i x_i} = (v_{,n} n_{,x_i} + v_{,\tau_\alpha} \tau_{\alpha,x_i})_{,x_i} \\ &= v_{,nn} n_{,x_i} n_{,x_i} + 2v_{,n\tau_\alpha} n_{,x_i} \tau_{\alpha,x_i} + v_{,\tau_\alpha \tau_\beta} \tau_{\alpha,x_i} \tau_{\beta,x_i} \\ &\quad + v_{,n} n_{,x_i x_i} + v_{,\tau_\alpha} \tau_{\alpha,x_i x_i} \equiv J. \end{aligned}$$

By the properties of curvilinear coordinates such that $\bar{n} \|\nabla n$, $\bar{\tau}_\alpha \|\nabla \tau_\alpha$, $\bar{n} \cdot \bar{\tau}_\alpha = 0$, $\bar{\tau}_\alpha \cdot \bar{\tau}_\beta = \delta_{\alpha\beta}$, $\alpha, \beta = 1, 2$, we have

$$\Delta v = J = v_{,nn} + v_{,\tau_\alpha \tau_\alpha} + v_{,n} \Delta n + v_{,\tau_\alpha} \Delta \tau_\alpha,$$

where we used that $n_{,x_i}n_{x_i} = 1$, $\tau_{\alpha,x_i}\tau_{\beta,x_i} = \delta_{\alpha\beta}$, $\alpha, \beta = 1, 2$.

Expressing the equation of continuity in the curvilinear coordinates we have

$$(3.32) \quad \operatorname{div} v \equiv v_{n,n} + v_n \operatorname{div} \bar{n} + v_{\tau_\alpha, \tau_\alpha} + v_{\tau_\alpha} \operatorname{div} \bar{\tau}_\alpha = 0,$$

where $v_n = v \cdot \bar{n}$, $v_{\tau_\alpha} = v \cdot \bar{\tau}_\alpha$, $\alpha = 1, 2$.

Next we formulate the second boundary condition (1.1)₅ in the curvilinear coordinates

$$(3.33) \quad v_{\tau_\alpha, n} - v_j \tau_{j\alpha, n} - v_i n_{i, \tau_\alpha} + \delta_{1j} \gamma v_{\tau_\alpha} = 0 \quad \text{on } S_j, \quad j = 1, 2, \quad \alpha = 1, 2.$$

Now, we calculate

$$(3.34) \quad \begin{aligned} \bar{n} \cdot \Delta v|_S &= (v_{n,nn} - \bar{n}_{,nn} \cdot v - 2\bar{n}_{,n} \cdot v_{,n} + v_{n, \tau_\alpha \tau_\alpha} \\ &\quad - \bar{n}_{, \tau_\alpha \tau_\alpha} \cdot v - 2\bar{n}_{, \tau_\alpha} \cdot v_{, \tau_\alpha} + \bar{n} \cdot v_{,n} \Delta n + \bar{n} \cdot v_{, \tau_\alpha} \Delta \tau_\alpha)|_S \\ &= (v_{n,nn} - \bar{n}_{,nn} \cdot v - 2\bar{n}_{,n} \cdot (v_n \bar{n} + v_{\tau_\alpha} \bar{\tau}_\alpha)_{,n} - \bar{n}_{, \tau_\alpha \tau_\alpha} \cdot v \\ &\quad - 2\bar{n}_{, \tau_\alpha} \cdot (v_n \bar{n} + v_{\tau_\beta} \bar{\tau}_\beta)_{, \tau_\alpha} + (v_{n,n} - \bar{n}_{,n} \cdot v) \Delta n \\ &\quad - \bar{n}_{, \tau_\alpha} \cdot v \Delta \tau_\alpha)|_S \\ &= (v_{n,nn} - \bar{n}_{,nn} \cdot v - 2\bar{n}_{,n} \cdot \bar{\tau}_\alpha v_{\tau_\alpha, n} - 2\bar{n}_{,n} \cdot \bar{\tau}_{\alpha, n} v_{\tau_\alpha} \\ &\quad - \bar{n}_{, \tau_\alpha \tau_\alpha} \cdot v - 2\bar{n}_{, \tau_\alpha} \cdot \bar{\tau}_\beta v_{\tau_\beta, \tau_\alpha} - 2\bar{n}_{, \tau_\alpha} \bar{\tau}_{\beta, \tau_\alpha} v_{\tau_\beta} \\ &\quad + (v_{n,n} - \bar{n}_{,n} \cdot v) \Delta n - \bar{n}_{, \tau_\alpha} \cdot v \Delta \tau_\alpha)|_S. \end{aligned}$$

From (3.32) we calculate

$$(3.35) \quad \begin{aligned} v_{n,nn} &= -v_{n,n} \operatorname{div} \bar{n} - v_{\tau_\alpha, \tau_\alpha n} - v_{\tau_\alpha, n} \operatorname{div} \bar{\tau}_\alpha \\ &\quad - v_n (\operatorname{div} \bar{n})_{,n} - v_{\tau_\alpha} (\operatorname{div} \bar{\tau}_\alpha)_{,n}. \end{aligned}$$

Projecting (3.35) on S and using (3.32) and (3.33) we obtain

$$(3.36) \quad \begin{aligned} v_{n,nn}|_{S_j} &= (v_{\tau_\alpha, \tau_\alpha} + v_{\tau_\alpha} \operatorname{div} \bar{\tau}_\alpha) \operatorname{div} \bar{n} - (v \cdot \bar{\tau}_{\alpha, n} + v \cdot \bar{n}_{, \tau_\alpha} - \delta_{1j} \gamma v_{\tau_\alpha})_{, \tau_\alpha} \\ &\quad - (v \cdot \bar{\tau}_{\alpha, n} + v \cdot \bar{n}_{, \tau_\alpha} - \delta_{1j} \gamma v_{\tau_\alpha}) \operatorname{div} \bar{\tau}_\alpha - v_{\tau_\alpha} (\operatorname{div} \bar{\tau}_\alpha)_{,n}, \quad j = 1, 2. \end{aligned}$$

Calculating $v_{n,nn}|_S$ from (3.36), $v_{n,n}|_S$ from (3.32) and $v_{\tau_\alpha, n}|_S$ from (3.33) and inserting them into (3.34) we obtain

$$(3.37) \quad \bar{n} \cdot \Delta v|_{S_j} = a_{j\alpha\beta} v_{\tau_\alpha, \tau_\beta} + a_j \cdot v, \quad j = 1, 2,$$

where for $S_j \in C^\alpha$ we have that $a_{j\alpha\beta} \in C^{\alpha-2}$, $a_j \in C^{\alpha-3}$, $j = 1, 2$.

From (3.30), (3.31) and (3.37) we obtain (3.29). This concludes the proof.

Now we estimate the norms $\|p\|_{L_\sigma(\Omega^t)}$, $\sigma = \frac{5}{3}, 2$. For this purpose we examine problem (3.29). Let G be the Green function to the Neumann problem (3.29). Then any solution to (3.29) can be expressed in the form (3.38)

$$p(x, t) = \int_{\Omega} G(x, y) [-\nabla \varrho \cdot v_t - \operatorname{div}(\varrho v \cdot \nabla v) + \operatorname{div}(\varrho f)] dx \\ - \sum_{j=1}^2 \int_{\check{S}_j} G(x, y) [-\varrho v \cdot \nabla v \cdot \bar{n} + \nu a_{j\alpha\beta} v_{\tau_\alpha} + \nu a_j \cdot v + \varrho f \cdot \bar{n}] dS_{jy}.$$

Integrating by parts in the second and the third expressions of the first integral and in the second expression of the second integral of (3.38) we get

$$(3.39) \quad p(x, t) = \int_{\Omega} [G(x, y)(-\nabla \varrho \cdot v_t) + (\varrho v \cdot \nabla v - \varrho f) \nabla_y G(x, y)] dy \\ - \sum_{j=1}^2 \int_{\check{S}_j} \{G(x, y) [-\nu a_{j\alpha\beta, \tau_\beta} v_{\tau_\alpha} + \nu a_j \cdot v] \\ - \nu G(x, y)_{, \tau_\beta} a_{j\alpha\beta} v_{\tau_\alpha}\} dS_{jy}.$$

Lemma 3.11. Assume that $v \in W_{5/3}^{2,1}(\Omega^T)$, $\varrho \in L_\infty(0, T; W_\infty^1(\Omega))$, $v \in V_2^0(\Omega^T)$, $f \in L_{5/3}(0, T; L_{15/14}(\Omega))$. Then the following inequality holds

$$(3.40) \quad \|p\|_{L_{5/3}(\Omega^T)} \leq \varepsilon \|v\|_{W_{5/3}^{2,1}(\Omega^T)} + c \varrho_x^* \|v_t\|_{L_{5/3}(\Omega^T)} \\ + c(1/\varepsilon, \varrho^*) d_3 + c \varrho^* \|f\|_{L_{5/3}(0, T; L_{15/14}(\Omega))},$$

where $\varepsilon \in (0, 1)$.

Proof. By the properties of the Green function we obtain

$$\|p\|_{L_{5/3}(\Omega)} \leq c[\varrho_x^* \|v_t\|_{L_{5/3}(\Omega)} + \varrho^* \|v \cdot \nabla v\|_{L_{\frac{15}{14}}(\Omega)} + \varrho^* \|f\|_{L_{\frac{15}{14}}(\Omega)} \\ + \|v\|_{W_{15/14}^{1-14/15}(S)}] \\ \leq c[\varrho_x^* \|v_t\|_{L_{5/3}(\Omega)} + \varrho^* \|v \cdot \nabla v\|_{L_{15/14}(\Omega)} \\ + \|v\|_{W_{15/14}^1(\Omega)} + \varrho^* \|f\|_{L_{15/14}(\Omega)}] \\ \leq c[\varrho_x^* \|v_t\|_{L_{5/3}(\Omega)} + \varrho^* \|v\|_{L_{30/13}(\Omega)} \|\nabla v\|_{L_2(\Omega)} \\ + \|v\|_{W_{15/14}^1(\Omega)} + \varrho^* \|f\|_{L_{15/14}(\Omega)}].$$

Integrating with respect to time we get

$$\begin{aligned} \|p\|_{L_{5/3}(\Omega^T)} &\leq c[\varrho_x^* \|v_t\|_{L_{5/3}(\Omega^T)} + \varrho^* \|v\|_{L_{10}(0,T;L_{\frac{30}{13}}(\Omega))} \|\nabla v\|_{L_2(\Omega^T)} \\ &\quad + \|v\|_{L_{5/3}(0,T;W_{15/14}^1(\Omega))} + \varrho^* \|f\|_{L_{5/3}(0,T';L_{15/14}(\Omega))}]. \end{aligned}$$

By certain interpolation and the energy type estimate we obtain

$$\begin{aligned} (3.41) \quad \|p\|_{L_{5/3}(\Omega^T)} &\leq \varepsilon \|v\|_{W_{5/3}^{2,1}(\Omega^T)} + c\varrho_x^* \|v_t\|_{L_{5/3}(\Omega^T)} \\ &\quad + c(1/\varepsilon, \varrho^*) \|v\|_{L_2(0,T;H^1(\Omega))} + c\varrho^* \|f\|_{L_{5/3}(0,T;L_{15/14}(\Omega))}. \end{aligned}$$

In view of (3.10) we get (3.40). This concludes the proof.

Next, we have

Lemma 3.12. *Assume that $v \in L_\infty(0, T; L_3(\Omega)) \cap V_2^0(\Omega^T)$, $v_t \in L_2(\Omega^T)$, $f \in L_2(0, T; L_{6/5}(\Omega))$.*

Then the following inequality is valid

$$\begin{aligned} (3.42) \quad \|p\|_{L_2(\Omega^T)} &\leq c\varrho_x^* \|v_t\|_{L_2(\Omega^T)} + c(\varrho^* \|v\|_{L_\infty(0,T;L_3(\Omega))} + 1)d_3 \\ &\quad + c\varrho^* \|f\|_{L_2(0,T;L_{6/5}(\Omega))}. \end{aligned}$$

Proof. By the properties of the Green function we have also

$$\begin{aligned} \|p\|_{L_2(\Omega)} &\leq c\varrho_x^* \|v_t\|_{L_2(\Omega)} + c\varrho^* (\|v \cdot \nabla v\|_{L_{6/5}(\Omega)} + \|f\|_{L_{6/5}(\Omega)}) \\ &\quad + c\|v\|_{W_{6/5}^1(\Omega)}. \end{aligned}$$

Integrating with respect to time is the L_2 -norm and using the Hölder inequality we obtain

$$\begin{aligned} (3.41) \quad \|p\|_{L_2(\Omega^T)} &\leq c\varrho_x^* \|v_t\|_{L_2(\Omega^T)} + c\varrho^* (\|v\|_{L_\infty(0,T;L_3(\Omega))} + 1) \|v\|_{L_2(0,T;H^1(\Omega))} \\ &\quad + c\varrho^* \|f\|_{L_2(0,T;L_{6/5}(\Omega))}. \end{aligned}$$

Using (3.10) we get (3.42). This concludes the proof.

4. Estimates

First we obtain an estimate for solutions to problem (3.19).

Lemma 4.1. *Assume that $\varrho \in L_\infty(0, T; W_\infty^1(\Omega))$, $\varrho_x^* = \|\varrho_x\|_{L_\infty(\Omega^T)}$, $\varrho \in C^\alpha(\Omega^T)$, $v' = (v_1, v_2)$, $v' \in L_5(\Omega^T) \cap L_\infty(0, T; L_2(\Omega)) \cap W_2^{1,1/2}(\Omega^T)$, $v'_t \in L_2(0, T; L_{6/5}(\Omega))$, $\nabla v' \in L_2(0, T; L_3(\Omega))$, $h \in L_\infty(0, T; L_3(\Omega))$, $F \in$*

$L_2(0, T; L_{6/5}(\Omega))$, $f' \in L_2(0, T; L_{6/5}(\Omega))$, $f' = (f_1, f_2)$, $\chi(0) \in L_2(\Omega)$. Assume that v is a weak solution to problem (1.1).

Then solutions to problem (3.19) satisfy the inequality

$$\begin{aligned}
(4.1) \quad & \sigma_1 \|\chi\|_{V_2^0(\Omega^t)} \leq c\varrho^* d_3 (\|h\|_{L_\infty(0,t;L_3(\Omega))} + \|F\|_{L_2(0,t;L_{6/5}(\Omega))}) \\
& + c\varrho_x^* (\|f'\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v_t'\|_{L_2(0,t;L_{6/5}(\Omega))}) \\
& + d_2 \|\nabla v'\|_{L_2(0,t;L_3(\Omega))} + c\varrho^* \|\chi(0)\|_{L_2(\Omega)} \\
& + \varphi(\varrho_*, \varrho^*, \varrho_x^*, \|\varrho\|_{\dot{C}^\alpha(\Omega^t)}) (\|v'\|_{L_5(\Omega^t)} + d_2 \\
& + \|v'\|_{W_2^{1,1/2}(\Omega^t)}), \quad t \leq T,
\end{aligned}$$

$\sigma_1 = \min\{\varrho_*, \nu\}$, φ is an increasing continuous positive function.

Proof. Multiplying (3.21)₁ by χ' , integrating over Ω , using the continuity equation (1.1)_{2,3} and the boundary conditions yields

$$\begin{aligned}
(4.2) \quad & \frac{d}{dt} \int_{\Omega} \varrho \chi'^2 dx + \nu \int_{\Omega} |\nabla \chi'|^2 dx = \int_{\Omega} \varrho h_3 \chi \chi' dx \\
& - \int_{\Omega} \varrho v \cdot \nabla \tilde{\chi} \chi' dx + \int_{\Omega} \varrho (F - v_{3,x_1} h_2 + v_{3,x_2} h_1) \chi' dx \\
& + \int_{\Omega} [\varrho_{x_1} (f_2 + v_{2,t} + v \cdot \nabla v_2) - \varrho_{x_2} (f_1 + v_{1,t} + v \cdot \nabla v_1)] \chi' dx.
\end{aligned}$$

Now we estimate the particular terms from the r.h.s. of (4.2). The first term we estimate by

$$\varepsilon \|\chi'\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \varrho^{*2} \|h_3\|_{L_3(\Omega)}^2 \|\chi\|_{L_2(\Omega)}^2,$$

the second we express in the form $\int_{\Omega} \varrho v \cdot \nabla \chi' \tilde{\chi} dx + \int_{\Omega} v \cdot \nabla \varrho \tilde{\chi} \chi' dx$ and estimate by

$$\varepsilon \|\nabla \chi'\|_{L_2(\Omega)}^2 + c(1/\varepsilon) (\varrho^{*2} + \|\nabla \varrho\|_{L_3(\Omega)}^2 \|v \tilde{\chi}\|_{L_2(\Omega)}^2),$$

the third by

$$\varepsilon \|\chi'\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \varrho^{*2} (\|F\|_{L_{6/5}(\Omega)}^2 + \|v_{3,x'}\|_{L_2(\Omega)}^2 \|h'\|_{L_3(\Omega)}^2),$$

where $h' = (h_1, h_2)$ and finally the last by

$$\varepsilon \|\chi'\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \varrho_x^{*2} (\|f'\|_{L_{6/5}(\Omega)}^2 + \|v_t'\|_{L_{6/5}(\Omega)}^2 + \|v\|_{L_2(\Omega)}^2 \|\nabla v'\|_{L_3(\Omega)}^2).$$

Using the above estimates in (4.2), assuming that ε is sufficiently small, integrating the result with respect to time and using Lemma 3.5 we obtain

$$\begin{aligned} \sigma_1 \|\chi'\|_{V_2^0(\Omega^t)}^2 &\leq c\varrho^{*2} d_3^2 (\|h\|_{L_\infty(0,t;L_3(\Omega))}^2 + \|\tilde{\chi}\|_{L_5(\Omega^t)}^2 \\ &\quad + \|F\|_{L_2(0,t;L_{6/5}(\Omega))}^2) + c\varrho_x^{*2} (\|f'\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + \|v_t'\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\ &\quad + d_3^2 \|\tilde{\chi}\|_{L_5(\Omega^t)}^2 + d_2^2 \|\nabla v'\|_{L_2(0,t;L_3(\Omega))}^2) + \varrho^{*2} \|\chi(0)\|_{L_2(\Omega)}^2. \end{aligned}$$

In view of the relation between χ and χ' we have

$$\begin{aligned} \sigma_1 \|\chi\|_{V_2^0(\Omega^t)}^2 &\leq c\varrho^{*2} d_3^2 (\|h\|_{L_\infty(0,t;L_3(\Omega))}^2 + \|F\|_{L_2(0,t;L_{6/5}(\Omega))}^2) \\ &\quad + c\varrho_x^{*2} (\|f'\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + \|v_t'\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\ (4.3) \quad &\quad + d_2^2 \|\nabla v'\|_{L_2(0,t;L_3(\Omega))}^2) + \varrho^{*2} \|\chi(0)\|_{L_2(\Omega)}^2 \\ &\quad + c(\varrho^{*2} + \varrho_x^{*2}) d_3^2 \|\tilde{\chi}\|_{L_5(\Omega^t)}^2 + \sigma_1 \|\tilde{\chi}\|_{V_2^0(\Omega^t)}^2. \end{aligned}$$

For solutions to problem (3.20) we obtain (see [Z6])

$$\begin{aligned} \|\tilde{\chi}\|_{L_5(\Omega^T)} &\leq \varphi(\varrho_*, \varrho^*, \|\varrho\|_{\dot{C}^\alpha(\Omega^T)}) \|v'\|_{L_5(\Omega^T)}, \\ (4.4) \quad \|\tilde{\chi}\|_{L_\infty(0,T;L_2(\Omega))} &\leq \varphi(\varrho_*, \varrho^*, \|\varrho\|_{\dot{C}^\alpha(\Omega^T)}) \|v'\|_{L_\infty(0,T;L_2(\Omega))}, \\ \|\nabla \tilde{\chi}\|_{L_2(\Omega^T)} &\leq \varphi(\varrho_*, \varrho^*, \|\varrho\|_{\dot{C}^\alpha(\Omega^T)}) \|v'\|_{W_2^{1,1/2}(\Omega^T)}, \end{aligned}$$

where φ is an increasing continuous positive function.

Using (4.4) in (4.3) implies (4.1). This concludes the proof.

Next we shall obtain an estimate for solutions to problem (3.17).

Lemma 4.2. *Assume that v is a weak solution to problem (1.1). Assume that $g \in L_2(0,T;L_{6/5}(\Omega))$, $h(0) \in L_2(\Omega)$, $f_3 \in L_2(0,T;L_{4/3}(S_2))$, $f \in L_2(0,T;L_{6/5}(\Omega))$, $v_t \in L_2(0,T;L_{6/5}(\Omega))$, $v \in L_\infty(0,T;L_3(\Omega))$, $\varrho \in L_\infty(0,T;W_\infty^1(\Omega))$.*

Assuming additionally that $h \in L_\infty(0,T;L_3(\Omega))$ we obtain

$$\begin{aligned} \sigma_1 \|h\|_{V_2^0(\Omega^t)}^2 &\leq c\varrho^{*2} (\|h\|_{L_\infty(0,t;L_3(\Omega))}^2 d_3^2 \\ (4.5) \quad &\quad + \|g\|_{L_2(0,t;L_{6/5}(\Omega))}^2) + c\varrho_x^{*2} (\|v_t\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\ &\quad + d_3^2 \|v\|_{L_\infty(0,t;L_3(\Omega))}^2 + \|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2) \\ &\quad + c(\|f_3\|_{L_2(0,t;L_{4/3}(S_2))}^2 + \|h(0)\|_{L_2(\Omega)}^2), \quad t \leq T, \end{aligned}$$

where σ_1 and ϱ_x^* are the same as in Lemma 4.1.

Replacing the condition $h \in L_\infty(0,T;L_3(\Omega))$ by $v \in L_2(0,T;W_3^1(\Omega))$ we have

$$\begin{aligned} \sigma_1 \|h\|_{V_2^0(\Omega^t)}^2 &\leq c \exp(\|\nabla v\|_{L_2(0,t;L_3(\Omega))}^2) \\ (4.6) \quad &\cdot [\|g\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + \|h(0)\|_{L_2(\Omega)}^2 + \varrho_x^{*2} (\|v_t\|_{L_2(0,t;L_{6/5}(\Omega))}^2 \\ &\quad + d_3^2 \|v\|_{L_\infty(0,t;L_3(\Omega))}^2 + \|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2 + \|f_3\|_{L_2(0,t;L_{4/3}(S_2))}^2)], \end{aligned}$$

where $t \leq T$.

Proof. Multiplying (3.17)₁ by h , integrating the result over Ω and using the continuity equation (1.1)₃ we obtain

$$(4.7) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho h^2 dx + \frac{\nu}{2} \|\mathbb{D}(h)\|_{L_2(\Omega)}^2 - \int_S \bar{n} \cdot \mathbb{T}(h, q) \cdot h dS \\ &= - \int_{\Omega} \varrho h \cdot \nabla v \cdot h dx + \int_{\Omega} \varrho g \cdot h dx - \int_{\Omega} \varrho_{,x_3} (v_t + v \cdot \nabla v - f) \cdot h dx. \end{aligned}$$

The boundary term in (4.7) equals

$$\frac{\gamma}{2} \int_{S_1} |h \cdot \bar{\tau}_\alpha|^2 dS_1 - \int_{S_2} q h_3 dS_2 \equiv I$$

so the second term in I is estimated by

$$\varepsilon \|h\|_{H^1(\Omega)}^2 + c(1/\varepsilon) \|f_3\|_{L_{\frac{4}{3}}(S_2)}^2.$$

The first term on the r.h.s. of (4.7) we estimate in two different ways. Either by

$$(4.8) \quad \varepsilon \|h\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \varrho^{*2} \|\nabla v\|_{L_2(\Omega)}^2 \|h\|_{L_3(\Omega)}^2$$

or by

$$(4.9) \quad \varepsilon \|h\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \varrho^* \int_{\Omega} \varrho h^2 dx \|\nabla v\|_{L_3(\Omega)}^2.$$

The second term on the r.h.s. of (4.7) we estimate by

$$\varepsilon \|h\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \varrho^{*2} \|g\|_{L_{6/5}(\Omega)}^2$$

and the last by

$$\varepsilon \|h\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \varrho_x^{*2} (\|v_t\|_{L_{6/5}(\Omega)}^2 + \|v \cdot \nabla v\|_{L_{6/5}(\Omega)}^2 + \|f\|_{L_{6/5}(\Omega)}^2).$$

Using the above estimates in (4.7), assuming that ε is sufficiently small, using Lemma 3.5 and integrating with respect to time we obtain (4.5) in the case (4.8) and (4.6) for (4.9). Let us mention that the time integral of the first term in I is deleted. This concludes the proof.

Let us consider the elliptic problem

$$(4.10) \quad \begin{aligned} v_{1,x_2} - v_{2,x_1} &= \chi & \text{in } \Omega', \\ v_{1,x_1} + v_{2,x_2} &= -h_3 & \text{in } \Omega', \\ v' \cdot \bar{n}|_{S'_1} &= 0. \end{aligned}$$

Let P be a plane perpendicular to the axis of the cylinder. Then $\Omega' = \Omega \cap P$, $S'_1 = S_1 \cap P$.

In view of (4.1) and (4.5) we obtain for solutions to problem (4.10) the inequality

$$(4.11) \quad \begin{aligned} \|v'\|_{V_2^1(\Omega^t)} &\leq \varphi(\varrho_*, \varrho^*, \|\varrho\|_{\dot{C}^\alpha(\Omega^T)}, d_3) [\|h\|_{L_\infty(0,t;L_3(\Omega))} \\ &+ \|v'\|_{L_5(\Omega^t)} + \|v'\|_{W_2^{1,1/2}(\Omega^t)} + G_1(t)] + c\varrho_x^* [d_2 \|\nabla v'\|_{L_2(0,t;L_3(\Omega))} \\ &+ \|v_t\|_{L_2(0,t;L_{6/5}(\Omega))} + \|\nabla v'\|_{L_2(0,t;L_3(\Omega))} \\ &+ \|v\|_{L_\infty(0,t;L_3(\Omega))} + \|f\|_{L_2(0,t;L_{6/5}(\Omega))}], \quad t \leq T, \end{aligned}$$

where

$$(4.12) \quad \begin{aligned} G_1(t) &= \|F\|_{L_2(0,t;L_{6/5}(\Omega))} + \|g\|_{L_2(0,t;L_{6/5}(\Omega))} \\ &+ \|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|\chi(0)\|_{L_2(\Omega)} + \|h(0)\|_{L_2(\Omega)}. \end{aligned}$$

Applying interpolation inequalities in (4.11) (see [BIN, Ch. 3, Sect. 10]) implies the inequality

$$(4.13) \quad \begin{aligned} \|v'\|_{V_2^1(\Omega^t)} &\leq \varphi(\varrho_*, \varrho^*, \|\varrho\|_{\dot{C}^\alpha(\Omega^T)}, d_3) [\|h\|_{L_\infty(0,t;L_3(\Omega))} \\ &+ \|v'\|_{L_2(\Omega;H^{1/2}(0,t))} + d_3 + G_1(t)] \\ &+ c\varrho_x^* [\|v_t\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v\|_{L_\infty(0,t;L_3(\Omega))} \\ &+ \|\nabla v'\|_{L_2(0,t;L_3(\Omega))} + \varphi(\varrho_x^*)d_3 \\ &+ \|f\|_{L_2(0,t;L_{6/5}(\Omega))}]. \end{aligned}$$

In view of (3.23) we have

$$(4.14) \quad \begin{aligned} \|v'\|_{V_2^1(\Omega^t)} &\leq \varphi(\varrho_*, \varrho^*, \varphi(T^{1/2}\|v\|_{W_2^{\sigma,\sigma/2}(\Omega^T)})X_1, d_3) \\ &\cdot [\|h\|_{L_\infty(0,t;L_3(\Omega))} + \|v'\|_{L_2(\Omega;W_2^{1/2}(0,t))} \\ &+ X_1(\|v_t\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v\|_{L_\infty(0,t;L_3(\Omega))} \\ &+ \|\nabla v'\|_{L_2(0,t;L_3(\Omega))}) + G_2(t)], \quad t \leq T, \end{aligned}$$

where $\sigma > \frac{5}{2}$,

$$(4.15) \quad G_2(t) = G_1(t) + \|f\|_{L_2(0,t;L_{6/5}(\Omega))} + d_3$$

and G_1 is defined by (4.12) and X_1 by (3.22).

Now, we consider problem (1.1) in the form

$$\begin{aligned}
(4.16) \quad & \varrho v_t - \operatorname{div} \mathbb{T}(v, p) = -\varrho v' \cdot \nabla v - \varrho v_3 h + \varrho f, \\
& \operatorname{div} v = 0 \\
& v \cdot \bar{n}|_S = 0 \\
& \bar{n} \cdot \mathbb{T}(v, p) \cdot \bar{\tau}_\alpha + \delta_{1j} \gamma v \cdot \bar{\tau}_\alpha|_{S_j} = 0, \quad \alpha = 1, 2, \quad j = 1, 2, \\
& v|_{t=0} = v(0).
\end{aligned}$$

Lemma 4.3. Assume that $f \in L_2(\Omega^T)$, $v_0 \in H^1(\Omega)$, $v \in W_2^{\sigma, \sigma/2}(\Omega^T)$, $\sigma > \frac{5}{2}$, $\varrho_* \leq \varrho \leq \varrho^*$, $\varrho_x(0) \in L_\infty(\Omega)$, $h \in L_\infty(0, T; L_3(\Omega)) \cap L_{\frac{10}{3}}(\Omega^T)$. Then for solutions to (4.16) the following inequality holds

$$\begin{aligned}
(4.17) \quad & \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \leq \varphi(\varrho_*, \varrho^*, \varphi(T^{1/2}\|v\|_{W_2^{\sigma, \sigma/2}(\Omega^T)})X_1, d_3) \\
& \cdot [H^2 + X_1^2(\|v_t\|_{L_2(\Omega^T)}^2 + \|v\|_{L_\infty(0, T; L_3(\Omega))}^2 + \|\nabla v'\|_{L_2(0, T; L_3(\Omega))}^2) + G^2] \\
& \equiv \varphi(H^2 + X_1^2 V^2 + G^2),
\end{aligned}$$

where X_1 is introduced in Lemma 3.9 (see (3.22)),

$$\begin{aligned}
(4.18) \quad & H = \|h\|_{L_\infty(0, T; L_3(\Omega))} + \|h\|_{L_{\frac{10}{3}}(\Omega^T)}, \\
& G = \|f\|_{L_2(\Omega^T)} + \|v_0\|_{H^1(\Omega)} + d_3 + \|F\|_{L_2(0, T; L_{6/5}(\Omega))} \\
& \quad + \|g\|_{L_2(0, T; L_{6/5}(\Omega))} + \|f_3\|_{L_2(0, T; L_{4/3}(S_2))}, \\
& V = \|v_t\|_{L_2(\Omega^T)} + \|v\|_{L_\infty(0, T; L_3(\Omega))} + \|\nabla v'\|_{L_2(0, T; L_3(\Omega))}
\end{aligned}$$

and φ is a generic function described by the r.h.s. of the above inequality.

Proof. From (2.5) (see Lemma 2.3), energy estimate (3.10),

$f' = -\varrho v' \cdot \nabla' v - \varrho v_3 h + \varrho f$ we obtain

$$\begin{aligned}
(4.19) \quad & \|v\|_{W_{5/3}^{2,1}(\Omega^T)} + \|\nabla p\|_{L_{5/3}(\Omega^T)} \\
& \leq \Phi[\|v\|_{L_{5/3}(\Omega^T)} + \|p\|_{L_{5/3}(\Omega^T)} + \varrho^* \|v'\|_{L_{10}(\Omega^T)} d_3 \\
& \quad + \varrho^* d_3 \|h\|_{L_{\frac{10}{3}}(\Omega^T)} + \varrho^* \|f\|_{L_{5/3}(\Omega^T)} + \|v_0\|_{W_{5/3}^{4/5}(\Omega)}],
\end{aligned}$$

where we used that

$$\|v' \nabla v\|_{L_{5/3}(\Omega^T)} \leq \|v'\|_{L_{10}(\Omega^T)} \|\nabla v\|_{L_2(\Omega^T)},$$

$$\|v_3 h\|_{L_{5/3}(\Omega^T)} \leq \|v_3\|_{L_{10/3}(\Omega^T)} \|h\|_{L_{10/3}(\Omega^T)}$$

and we introduced the quantity

$$(4.20) \quad \Phi = \varphi(\varrho_*, \varrho^*, \varphi(T^{1/2}\|v\|_{W_2^{\sigma, \sigma/2}(\Omega^T)})X_1, d_3).$$

In view of the imbedding (see [Z3, Lemma 3.7])

$$(4.21) \quad \|v'\|_{L_{10}(\Omega^T)} \leq c\|v'\|_{V_2^1(\Omega^T)},$$

notation (4.18) we obtain, from (4.19) after some interpolations, the inequality

$$(4.22) \quad \begin{aligned} & \|v\|_{W_{5/3}^{2,1}(\Omega^T)} + \|\nabla p\|_{L_{5/3}(\Omega^T)} \\ & \leq \Phi[\|p\|_{L_{5/3}(\Omega^T)} + H + X_1(\|v_t\|_{L_2(0,T;L_{6/5}(\Omega))} + \|v\|_{L_\infty(0,T;L_3(\Omega))} \\ & \quad + \|\nabla v'\|_{L_2(0,T;L_3(S_2))}) + G_3], \end{aligned}$$

where

$$(4.23) \quad G_3 = G_2 + \|f\|_{L_{5/3}(\Omega^T)} + \|v_0\|_{W_{5/3}^{4/5}(\Omega)}$$

and G_2 is defined by (4.15).

In view of (3.40) we obtain from (4.22) the inequality

$$(4.24) \quad \begin{aligned} & \|v\|_{W_{5/3}^{2,1}(\Omega^T)} + \|\nabla p\|_{L_{5/3}(\Omega^T)} \leq \Phi[H + X_1(\|v_t\|_{L_{5/3}(\Omega^T)} \\ & \quad + \|v\|_{L_\infty(0,T;L_3(\Omega))} + \|\nabla v'\|_{L_2(0,T;L_3(\Omega))}) + G_3]. \end{aligned}$$

Using (4.24) in (4.14) yields

$$(4.25) \quad \begin{aligned} & \|v'\|_{V_2^1(\Omega^T)} \leq \Phi[H + X_1(\|v_t\|_{L_{5/3}(\Omega^T)} \\ & \quad + \|v\|_{L_\infty(0,T;L_3(\Omega))} + \|\nabla v'\|_{L_2(0,T;L_3(\Omega))}) + G_3]. \end{aligned}$$

From (2.5) (see Lemma 2.3) we have

$$(4.26) \quad \begin{aligned} & \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \\ & \leq \Phi[T^{1/6}\|v\|_{L_2(\Omega^T)} + \|p\|_{L_2(\Omega^T)} + \varrho^*\|v' \cdot \nabla v\|_{L_2(\Omega^T)} \\ & \quad + \varrho^*\|v_3 h\|_{L_2(\Omega^t)} + \varrho^*\|f\|_{L_2(\Omega^T)} + \|v_0\|_{H^1(\Omega)}], \end{aligned}$$

where the third and the fourth terms we estimate by

$$(4.27) \quad \begin{aligned} & \|v' \cdot \nabla v\|_{L_2(\Omega^T)} \leq \|v'\|_{L_{10}(\Omega^t)}\|v\|_{W_{5/3}^{2,1}(\Omega^T)}, \\ & \|v_3 h\|_{L_2(\Omega^T)} \leq \|v_3\|_{W_{5/3}^{2,1}(\Omega^T)}\|h\|_{L_{10/3}(\Omega^t)}. \end{aligned}$$

Using the energy estimate (3.10), (4.27), (4.24), (4.25) and (3.41) in (4.26) we obtain

$$(4.28) \quad \begin{aligned} & \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \\ & \leq \Phi[H^2 + X_1^2(T^{1/6}\|v_t\|_{L_2(\Omega^T)}^2 + \|v\|_{L_\infty(0,T;L_3(\Omega))}^2 \\ & \quad + \|\nabla v'\|_{L_2(0,T;L_3(S_2))}) + G_4^2], \end{aligned}$$

where we used the estimate

$$\|v_t\|_{L_{5/3}(\Omega^T)} \leq |\Omega|^{1/6} T^{1/6} \|v_t\|_{L_2(\Omega^T)}$$

and

$$(4.29) \quad G_4 = G_3 + \|f\|_{L_2(\Omega^T)} + \|v_0\|_{H^1(\Omega)},$$

where G_3 is defined by (4.23).

Since $G_4 \leq cG$ we obtain from (4.28) inequality (4.17). This concludes the proof.

Now we increase regularity from Lemma 4.3 up to $v \in W_2^{2+s,1+s/2}(\Omega^T)$, $s+2 \geq \sigma > \frac{5}{2}$.

Lemma 4.4. *Assume that $s \in (\frac{1}{2}, 1)$, v is a weak solution to problem (1.1), $\varrho_* \leq \varrho_0 \leq \varrho^*$, $\varrho_x(0) \in L_\infty(\Omega)$, $v \in W_2^{2+s,1+s/2}(\Omega^T)$, $h \in L_\infty(0, T; L_3(\Omega)) \cap L_{\frac{10}{3}}(\Omega^T)$, $f \in W_2^{s,s/2}(\Omega^T)$, $v_0 \in W_2^{1+s}(\Omega)$, $f \in L_2(0, T; W_{6/5}^1(\Omega))$. Then*

$$(4.30) \quad \begin{aligned} & \|v\|_{W_2^{2+s,1+s/2}(\Omega^T)} + \|\nabla p\|_{W_2^{s,s/2}(\Omega^T)} \\ & \leq \varphi(\varrho_*, \varrho^*, \varphi(T^{1/2}\|v\|_{H^{2+s,1+s/2}(\Omega^T)})X_1, d_3)[\varphi(\|v\|_{H^{2+s,1+s/2}(\Omega^T)})X_2 \\ & \quad + \|v\|_{H^{2+s,1+s/2}(\Omega^T)}X_1 + \varphi(H + \|v\|_{H^{2+s,1+s/2}(\Omega^T)}X_1 + G, d_3) \\ & \quad + K], \end{aligned}$$

where

$$(4.31) \quad \begin{aligned} K &= \|f\|_{W_2^{s,s/2}(\Omega^T)} + \|v_0\|_{W_2^{1+s}(\Omega)} + d_3, \\ X_2 &= X_1 + \|\varrho_{xx}(0)\|_{L_q(\Omega)} + \|\partial_t^{s/2}\varrho_x(0)\|_{L_r(\Omega)}, \end{aligned}$$

where $q \leq \frac{3}{\frac{3}{2}-s}$, $s \in (1/2, 1)$, $3/2 - s \leq 3/q \leq 1/2 + 3/r$, $r \leq 6$, G, H are defined by (4.18).

Proof. From (2.6) we have

$$(4.32) \quad \begin{aligned} & \|v\|_{H^{s+2,s/2+1}(\Omega^T)} + \|\nabla p\|_{H^{s,s/2}(\Omega^T)} \\ & \leq \Phi[d_3 + \|p\|_{H^{s,s/2}(\Omega^T)} + \|\varrho v \cdot \nabla v\|_{H^{s,s/2}(\Omega^T)} + \|\varrho f\|_{H^{s,s/2}(\Omega^T)} \\ & \quad + \|v_0\|_{H^{1+s}(\Omega)}], \end{aligned}$$

where Φ is defined by (4.20).

Since $\varrho_x, \varrho_t \in L_\infty(\Omega^T)$ and since we are interested in the case $s < 1$ we have

$$(4.33) \quad \begin{aligned} \|\varrho v \cdot \nabla v\|_{H^{s,s/2}(\Omega^T)} &\leq (\varrho_x^* + \varrho_t^*) \|v \cdot \nabla v\|_{L_2(\Omega^T)} \\ &\quad + \varrho^* \|v \cdot \nabla v\|_{H^{s,s/2}(\Omega^T)}. \end{aligned}$$

To estimate the last term in (4.33) it is sufficient to examine the highest order terms. First we use the splitting

$$(4.34) \quad \|v \cdot \nabla v\|_{H^{s,s/2}(\Omega)} = \|v \cdot \nabla v\|_{L_2(0,T;H^s(\Omega))} + \|v \cdot \nabla v\|_{L_2(\Omega;H^{s/2}(0,T))}.$$

It is sufficient to examine only one norm. Therefore, we consider

$$\|v \cdot \nabla v\|_{L_2(0,T;H^s(\Omega))} = \|\partial_x^s v \nabla v\|_{L_2(\Omega^T)} + \|v \cdot \nabla \partial_x^s v\|_{L_2(\Omega^T)} + \|v \cdot \nabla v\|_{L_2(\Omega^T)}.$$

Hence we examine only the first two norms. By the Hölder inequality we have

$$(4.35) \quad \begin{aligned} \|\partial_x^s v \cdot \nabla v\|_{L_2(\Omega^T)} &\leq \|\partial_x^s v\|_{L_5(\Omega^T)} \|\nabla v\|_{L_{10/3}(\Omega^T)} \\ &\leq (\varepsilon_1^{1-\varkappa_1} \|v\|_{H^{2+s,1+s/2}(\Omega^T)} + c\varepsilon_1^{-\varkappa_1} \|v\|_{L_2(\Omega^T)}) \|v\|_{H^{2,1}(\Omega^T)}, \end{aligned}$$

where $\varkappa_1 = \frac{\frac{3}{2}+s}{2+s} < 1$ and there is no restrictions on $s \in (1/2, 1)$.

Similarly, we have

$$(4.36) \quad \begin{aligned} \|v \cdot \nabla \partial_x^s v\|_{L_2(\Omega^T)} &\leq \|v\|_{L_{10}(\Omega^T)} \|\nabla \partial_x^s v\|_{L_{5/2}(\Omega^T)} \\ &\leq [\varepsilon_2^{1-\varkappa_1} \|v\|_{H^{2+s,1+s/2}(\Omega^T)} + c\varepsilon_2^{-\varkappa_1} \|v\|_{L_2(\Omega^T)}] \|v\|_{H^{2,1}(\Omega^T)}. \end{aligned}$$

Next

$$(4.37) \quad \|\varrho f\|_{H^{s,s/2}(\Omega^T)} \leq (\varrho_x^* + \varrho_t^* + \varrho^*) \|f\|_{H^{s,s/2}(\Omega^T)}.$$

Finally, we examine the term with pressure. We have

$$(4.38) \quad \|p\|_{H^{s,s/2}(\Omega^T)} = \|p\|_{L_2(\Omega;H^{s/2}(0,T))} + \|p\|_{L_2(0,T;H^s(\Omega))}.$$

Applying $\partial_t^{s/2}$ to (3.39) and integrating the result over Ω^t yields

$$\begin{aligned}
\|\partial_t^{s/2} p\|_{L_2(\Omega^t)} &\leq c \left[\int_0^t (\|\partial_t^{s/2}(\nabla \varrho v_t)\|_{L_{1'}(\Omega)}^2 \right. \\
&\quad \left. + \|\partial_t^{s/2}(\varrho v \cdot \nabla v - \varrho f)\|_{L_{6/5}(\Omega)}^2 + \|\partial_t^{s/2} v\|_{W_{6/5}^1(\Omega)}^2) dt \right]^{1/2} \\
&\leq c \left[\int_0^t (\|\partial_t^{s/2} \nabla \varrho\|_{L_{2'}(\Omega)}^2 \|v_t\|_{L_2(\Omega)}^2 + \|\varrho_x\|_{L_{2'}(\Omega)}^2 \|\partial_t^{s/2} v_t\|_{L_2(\Omega)}^2 \right. \\
&\quad + \|\partial_t^{s/2} \varrho v \nabla v\|_{L_{6/5}(\Omega)}^2 + \|\varrho \partial_t^{s/2} v \nabla v\|_{L_{6/5}(\Omega)}^2 \\
&\quad + \|\varrho v \partial_t^{s/2} \nabla v\|_{L_{6/5}(\Omega)}^2 + \|\partial_t^{s/2} \varrho f\|_{L_{6/5}(\Omega)}^2 \\
&\quad \left. + \|\varrho \partial_t^{s/2} f\|_{L_{6/5}(\Omega)}^2 + \|\partial_t^{s/2} \nabla v\|_{L_{6/5}(\Omega)}^2 + \|\partial_t^{s/2} v\|_{L_{6/5}(\Omega)}^2) dt \right]^{1/2} \\
&\leq c \{ \sup_t \|\partial_t^{s/2} \nabla \varrho\|_{L_{2'}(\Omega)} \|v_t\|_{L_2(\Omega^t)} + \varrho_x^* \|\partial_t^{s/2} v_t\|_{L_2(\Omega^t)} \\
&\quad + (\varrho_x^* + \varrho_t^*) [\sup_t \|v\|_{L_2(\Omega)} \|\nabla v\|_{L_2(0,t;L_3(\Omega))} + \|f\|_{L_2(0,t;L_{6/5}(\Omega))}] \\
&\quad + \varrho^* (\sup_t \|\partial_t^{s/2} v\|_{L_3(\Omega)} \|\nabla v\|_{L_2(\Omega^t)} + \sup_t \|v\|_{L_2(\Omega)} \|\partial_t^{s/2} \nabla v\|_{L_2(0,t;L_3(\Omega))} \\
&\quad + \|\partial_t^{s/2} f\|_{L_2(0,t;L_{6/5}(\Omega))}) + \|\partial_t^{s/2} \nabla v\|_{L_2(0,t;L_{6/5}(\Omega))} \\
&\quad \left. + \|\partial_t^{s/2} v\|_{L_2(0,t;L_{6/5}(\Omega))} \right\} \equiv I_1,
\end{aligned}$$

where $1' > 1$, $2' > 2$ but arbitrary close to 1 and 2, respectively.

Using interpolation inequalities (see [BIN, Ch. 3]) and the estimate for the weak solution we get

$$\begin{aligned}
\|\partial_t^{s/2} p\|_{L_2(\Omega^t)} &\leq I_1 \leq c(\sup_t \|\partial_t^{s/2} \varrho_x\|_{L_{2'}(\Omega)} + \varrho_x^* \\
(4.39) \quad &+ \varrho_t^* + \varepsilon) V_s(t) + \varphi(1/\varepsilon, d_3, \varrho^*)(d_3 + \|f\|_{L_2(0,t;L_{6/5}(\Omega))} \\
&+ \|\partial_t^{s/2} f\|_{L_2(\Omega^t)}),
\end{aligned}$$

where we used the notation

$$V_s(T) = \|v\|_{H^{2+s, 1+s/2}(\Omega^T)}.$$

To estimate the norm $\sup_t \|\partial_t^{s/2} \varrho_x\|_{L_r(\Omega)}$ we differentiate (1.1)₃ with respect to $\partial_t^{s/2} \partial_x$, multiply by $\partial_t^{s/2} \partial_x \varrho |\partial_t^{s/2} \partial_x \varrho|^{r-2}$, use (1.1)_{2,4} and integrate over Ω . Then we obtain

$$\begin{aligned}
(4.40) \quad \frac{d}{dt} \|\partial_t^{s/2} \varrho_x\|_{L_r(\Omega)} &\leq \left(\int_{\Omega} |\partial_t^{s/2} v|^r |\varrho_{xx}|^r dx \right)^{1/r} \\
&+ \varrho_x^* \|\partial_t^{s/2} v_x\|_{L_r(\Omega)} + \|v_x\|_{L_{\infty}(\Omega)} \|\partial_t^{s/2} \varrho_x\|_{L_r(\Omega)}.
\end{aligned}$$

Integrating (4.40) with respect to time yields

$$\begin{aligned}
(4.41) \quad & \|\partial_t^{s/2} \varrho_x(t)\|_{L_r(\Omega)} \leq \exp \left(\int_0^t \|v_x(t')\|_{L_\infty(\Omega)} dt' \right) \\
& \cdot \left[\int_0^t \left(\int_\Omega |\partial_{t'}^{s/2} v|^r |\varrho_{xx}|^r dx \right)^{1/r} dt' \right. \\
& \left. + \varrho_x^* \int_0^t \|\partial_t^{s/2} v_x\|_{L_r(\Omega)} dt' + \|\partial_t^{s/2} \varrho_x(0)\|_{L_r(\Omega)} \right].
\end{aligned}$$

On the r.h.s. of (4.41) some norm of ϱ_{xx} appears. To estimate it we differentiate (1.1)₃ twice with respect to x , multiply by $\varrho_{xx}|\varrho_{xx}|^{q-2}$, use (1.1)_{2,4} and integrate over Ω . Then we obtain

$$\frac{d}{dt} \|\varrho_{xx}\|_{L_q(\Omega)} \leq \|v_x\|_{L_\infty(\Omega)} \|\varrho_{xx}\|_{L_q(\Omega)} + \varrho_x^* \|v_{xx}\|_{L_q(\Omega)}.$$

Integrating the inequality with respect to time yields

$$\begin{aligned}
(4.42) \quad & \|\varrho_{xx}(t)\|_{L_q(\Omega)} \leq \exp \left(\int_0^t \|v_x(t')\|_{L_\infty(\Omega)} dt' \right) \\
& \cdot \left[\varrho_x^* \int_0^t \|v_{xx}(t')\|_{L_q(\Omega)} dt' + \|\varrho_{xx}(0)\|_{L_q(\Omega)} \right].
\end{aligned}$$

Now we have to determine r, q in (4.37), (4.38), respectively.

Looking for $v \in H^{s+2, s/2+1}(\Omega^T)$ we see that $v_{xx} \in L_2(0, T; L_q(\Omega))$ with $q \leq \frac{3}{\frac{3}{2}-s}$.

Hence, (4.42) implies that $\varrho_{xx} \in L_\infty(0, T; L_q(\Omega))$, $q \leq 3/(3/2 - s)$. Estimating the first term under the square bracket in (4.41) by

$$\begin{aligned}
& \|\varrho_{xx}\|_{L_\infty(0, T; L_{r\lambda_1}(\Omega))} \int_0^t \|\partial_t^{s/2} v\|_{L_{r\lambda_2}(\Omega)} dt' \\
& \leq c(T) \|\varrho_{xx}\|_{L_\infty(0, T; L_q(\Omega))} \|v\|_{H^{s+2, s/2+1}(\Omega^T)},
\end{aligned}$$

we need $1/\lambda_1 + 1/\lambda_2 = 1$, $q = r\lambda_1$, $\frac{3}{2} - \frac{3}{r\lambda_2} \leq 2$.

Hence

$$\frac{3}{2} - s \leq \frac{3}{r\lambda_1} \leq \frac{1}{2} + \frac{3}{r}$$

which implies the restrictions

$$(4.43) \quad 1 \leq s + \frac{3}{r}, \quad r \leq q.$$

In view of the above considerations we express (4.42) in the form

$$(4.44) \quad \begin{aligned} \|\varrho_{xx}(t)\|_{L_q(\Omega)} &\leq \exp(\|v_x\|_{L_1(0,t;L_\infty(\Omega))}) \\ &\cdot [\varrho_x^* t^{1/2} \|v\|_{H^{2+s,1+s/2}(\Omega^t)} + \|\varrho_{xx}(0)\|_{L_q(\Omega)}], \end{aligned}$$

where

$$(4.45) \quad q \leq \frac{3}{\frac{3}{2} - s}.$$

Using the notation

$$(4.46) \quad V_s(T) = \|v\|_{H^{2+s,1+s/2}(\Omega^T)}$$

we have

$$(4.47) \quad \|v_x\|_{L_1(0,T;L_\infty(\Omega))} \leq cT^{1/2}V_s(T), \quad \text{for } s > \frac{1}{2}.$$

Using (4.44) in (4.41) and using that the second term under the square bracket in the r.h.s. of (4.41) is estimated by

$$c(t)V_s(t)$$

under the assumption

$$(4.48) \quad r \leq 6,$$

we obtain from (4.41) the inequality

$$(4.49) \quad \begin{aligned} \|\partial_t^{s/2}\varrho_x(t)\|_{L_r(\Omega)} &\leq \exp(t^{1/2}V_s(t))[\exp(t^{1/2}V_s(t)) \\ &\cdot (\varrho_x^* t^{1/2}V_s + \|\varrho_{xx}(0)\|_{L_q(\Omega)})t^{1/2}V_s(t) \\ &+ \varrho_x^* t^{1/2}V_s(t) + \|\partial_t^{s/2}\varrho_x(0)\|_{L_r(\Omega)}] \\ &\leq \varphi(t^{1/2}V_s(t))[\varrho_x^* t^{1/2}V_s(t) + \|\varrho_{xx}(0)\|_{L_q(\Omega)} + \|\partial_t^{s/2}\varrho_x(0)\|_{L_r(\Omega)}], \end{aligned}$$

where $3/2 - s \leq 3/q \leq 1/2 + 3/r$.

Employing (4.49), (3.25) and (3.26) in (4.39) we obtain

$$(4.50) \quad \begin{aligned} \|\partial_t^{s/2}p\|_{L_2(\Omega^t)} &\leq \varphi(t^{1/2}V_s(t)X_1, \varrho_*, \varrho^*, d_3) \\ &\cdot [(X_2 + \varepsilon)t^{1/2}V_s(t) + \varphi(1/\varepsilon, d_3, \varrho_*, \varrho^*)(\|f\|_{L_2(0,t;L_{6/5}(\Omega))} \\ &+ \|\partial_t^{s/2}f\|_{L_2(\Omega^t)})]. \end{aligned}$$

Now we examine $\|p\|_{L_2(0,t;H^s(\Omega))}$. Applying ∂_x^s to (3.39) and taking the $L_2(\Omega)$ norm we obtain

$$\begin{aligned}
(4.51) \quad & \|\partial_x^s p\|_{L_2(\Omega)} \leq c(\|p_x\|_{L_2(\Omega)} + \|p\|_{L_2(\Omega)}) \leq c(\|\nabla \varrho v_t\|_{L_{6/5}(\Omega)} \\
& + \|\varrho v \nabla v\|_{L_2(\Omega)} + \|\varrho f\|_{L_2(\Omega)} + \|v\|_{W_2^{2-1/2}(S)}) \\
& \leq c\varrho_x^* \|v_t\|_{L_{6/5}(\Omega)} + c\varrho^* \|v\|_{L_3(\Omega)} \|\nabla v\|_{L_2(\Omega)} \\
& + c\varrho^* \|f\|_{L_2(\Omega)} + c\|v\|_{W_2^2(\Omega)}.
\end{aligned}$$

We obtain from (4.51) after integration with respect to time the inequality

$$\begin{aligned}
(4.52) \quad & \|p_{,x}\|_{L_2(\Omega^t)} \leq c\varrho_x^* V_s(t) + \varphi(d_3, \varrho_x^*)(\varepsilon V_s(t) \\
& + c(1/\varepsilon)d_3 + \|f\|_{L_2(\Omega^t)}).
\end{aligned}$$

Using (4.33)–(4.37), (4.50) and (4.52) in (4.32) implies the inequality

$$\begin{aligned}
(4.53) \quad & \|v\|_{H^{2+s,1+s/2}(\Omega^T)} + \|\nabla p\|_{H^{s,s/2}(\Omega^T)} \\
& \leq \varphi(T^{1/2}V_s(T)X_1, \varrho_*, \varrho^*, d_3)[(X_2 + \varepsilon)T^{1/2}V_s(T) \\
& + \varphi(1/\varepsilon, d_3)(\|v\|_{H^{2,1}(\Omega^T)} + \|f\|_{H^{s,s/2}(\Omega^T)}) + \varphi(1/\varepsilon, \varrho_*, \varrho^*, d_3)].
\end{aligned}$$

For sufficiently small ε and (4.17) we obtain (4.30). This concludes the proof.

Remark 4.5. In formulas (3.25), (3.26), (4.41) and (4.44) we have the expression

$$I(t) = \exp \int_0^t \|v_x(t')\|_{L_\infty(\Omega)} dt'.$$

In view of the imbedding

$$\|v\|_{L_2(0,T;W_\infty^1(\Omega))} \leq cV_s(T), \quad s > \frac{1}{2}$$

we obtain the estimate

$$I(t) \leq \exp(ct^{1/2}V_s(t)), \quad t \leq T,$$

which is not convenient because factor $t^{1/2}$ appears under the exponent functions. The difficulty can be cancelled in virtue of the assumption

$$1 \leq \|v\|_{W_\infty^1(\Omega)}.$$

The above inequality is not restrictive because the case $\|v\|_{W_\infty^1(\Omega)} \leq 1$ implies in view of Lemma 3.5 the following estimate for solutions to problem (1.1)

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} &\leq \varphi(\varrho_*, \varrho^*, d_3, e^{cT}) \\ &\cdot [\|f\|_{L_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}], \quad c > 0. \end{aligned}$$

Hence regularity $H^{s+2, s/2+1}(\Omega^T)$ follows immediately. In this case the constant in the above estimate depends on time but this does not imply any restrictions on magnitudes of the data.

Now we pass to problem (3.17). Then we have

Lemma 4.6. *Assume that $\frac{1}{2} < \sigma \leq s < 1$ and σ can be chosen as very close to s .*

Let us take Remark 4.5 under account and let

$$(4.54) \quad \begin{aligned} B &= \|g\|_{L_2(0,T;L_{6/5}(\Omega))} + \|f_3\|_{L_2(0,T;L_{4/3}(S_2))} \\ &+ \|h(0)\|_{L_2(\Omega)} < \infty, \end{aligned}$$

be sufficiently small.

Let $f, g \in L_2(\Omega^T)$, $h(0) \in H^1(\Omega)$, $v \in H^{2+\sigma, 1+\sigma/2}(\Omega^T)$ and let (v, ϱ) be the weak solution to (1.1) described by Lemma 3.5. Then solutions to (3.17) satisfy the inequality

$$(4.55) \quad \begin{aligned} \|h\|_{H^{2,1}(\Omega^T)} + \|\nabla q\|_{L_2(\Omega^T)} &\leq \varphi(\varrho_*, \varrho^*, \varphi(V_\sigma(T))X_1, d_3) \\ &\cdot [\varphi(V_\sigma(T))(X_1 + B) + K_0(T)], \end{aligned}$$

where X_1 is introduced in (3.22) and

$$(4.56) \quad K_0(T) = \|g\|_{L_2(\Omega^T)} + \|f\|_{L_2(\Omega^T)} + \|h(0)\|_{H^1(\Omega)}.$$

Proof. For solutions to (3.17) we have

$$(4.57) \quad \begin{aligned} \|h\|_{H^{2,1}(\Omega^T)} + \|\nabla q\|_{L_2(\Omega^T)} &\leq \varphi(\varrho_*, \varrho^*, \varphi(V_\sigma(T))X_1, d_3) \\ &\cdot [\|v \cdot \nabla h\|_{L_2(\Omega^T)} + \|h \cdot \nabla v\|_{L_2(\Omega^T)} + \|g\|_{L_2(\Omega^T)} \\ &+ \varphi(V_\sigma(T))X_1 (\|v_t\|_{L_2(\Omega^T)} + \|v \cdot \nabla v\|_{L_2(\Omega^T)} + \|f\|_{L_2(\Omega^T)}) \\ &+ \|h(0)\|_{H^1(\Omega)}]. \end{aligned}$$

We need the inequalities

$$(4.58) \quad \begin{aligned} \|v \cdot \nabla h\|_{L_2(\Omega^T)} &\leq \|\nabla h\|_{L_3(\Omega^T)} \|v\|_{L_6(\Omega^T)} \\ &\leq \varepsilon \|h\|_{H^{2,1}(\Omega^T)} + \varphi(\|v\|_{W_2^{2,1}(\Omega^T)}) \|h\|_{L_2(\Omega^T)}, \end{aligned}$$

$$\begin{aligned}
(4.59) \quad \|h \cdot \nabla v\|_{L_2(\Omega^T)} &\leq \|h\|_{L_6(\Omega^T)} \|\nabla v\|_{L_3(\Omega^T)} \\
&\leq \varepsilon \|h\|_{H^{2,1}(\Omega^T)} + \varphi(\|v\|_{W_2^{2,1}(\Omega^T)}) \|h\|_{L_2(\Omega^T)}
\end{aligned}$$

$$(4.60) \quad \|v \cdot \nabla v\|_{L_2(\Omega^T)} \leq \|v\|_{L_\infty(\Omega^T)} \|\nabla v\|_{L_2(\Omega^T)} \leq d_3 \|v\|_{H^{2+\sigma, 1+\sigma/2}(\Omega^T)}.$$

In view of (4.6) we have

$$\begin{aligned}
(4.61) \quad \|h\|_{L_2(\Omega^T)} &\leq \varphi(\varrho_*, \varrho^*, d_3) \exp(V_\sigma(T)) [B \\
&\quad + X_1(V_\sigma(T) + \|f\|_{L_2(0,T;L_{4/3}(S_2))})].
\end{aligned}$$

Employing (4.58)–(4.61) in (4.57) implies (4.55). This concludes the proof. Finally, we have

Theorem 4.7. *Let assumptions of Lemmas 4.4, 4.6 hold. Let us take Remark 4.5 under account. Let*

$$(4.68) \quad X = X_1 + X_2 + B$$

Then for sufficiently small X the estimate holds

$$(4.69) \quad V_s(T) \leq \varphi(\varrho_*, \varrho^*, G, K, K_0, X, L), \quad \frac{1}{2} < s < 1,$$

where K_0 is introduced in (4.56), K in (4.31), G in (4.18) and

$$(4.70) \quad L = \|f_3\|_{L_2(0,T;H^{1/2}(S))} + \|f\|_{H^{s,s/2}(\Omega^T)}.$$

Proof. Applying a fixed point argument in (4.30) we obtain for sufficiently small X_2 the inequality

$$(4.71) \quad V_s(T) \leq \varphi(\varrho_*, \varrho^*, \|h\|_{H^{2,1}(\Omega^T)}, G, K, X_2).$$

Using (4.55) with $\sigma = s$ in (4.71) and applying again a fixed point argument for sufficiently small X we obtain (4.69). This concludes the proof.

5. Existence

We prove the existence of solutions to problem (1.1) by the Leray-Schauder fixed point theorem. For this purpose we construct a mapping Φ in the following way.

Let $\tilde{v} \in W_2^{2+\sigma, 1+\sigma/2}(\Omega^T)$, $\operatorname{div} \tilde{v} = 0$, $\tilde{v} \cdot \bar{n}|_S = 0$, $\sigma \in (1/2, 1)$ be given.

Then $\varrho = \varrho(\tilde{v})$ is a solution to the problem

$$(5.1) \quad \varrho_t + \tilde{v} \cdot \nabla \varrho = 0, \quad \varrho|_{t=0} = \varrho_0.$$

Let

$$(5.2) \quad v = \Phi(\tilde{v}, \lambda)$$

be a solution to the problem

$$(5.3) \quad \begin{aligned} & [\varrho_0(1 - \lambda) + \lambda \varrho(\tilde{v})]v_t - \operatorname{div} \mathbb{T}(v, p) \\ &= \lambda[-\varrho(\tilde{v})\tilde{v} \cdot \nabla \tilde{v} + \varrho(\tilde{v})f] \\ & \operatorname{div} v = 0 \\ & v \cdot \bar{n}|_S = 0, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha|_S = 0, \quad \alpha = 1, 2, \\ & v|_{t=0} = v_0, \end{aligned}$$

where $\lambda \in [0, 1]$.

In view of Lemma 4.4, Remark 4.5, Lemma 4.6 and Theorem 4.7 we have

Lemma 5.1. *Assume that $v_0 \in H^{1+s}(\Omega)$, $v_{0,x_3} \in H^1(\Omega)$, $f \in H^{s,s/2}(\Omega^T) \cap L_2(0, T; W_{6/5}^1(\Omega))$, $f_{,x_3} \in L_2(\Omega^T)$, $s \in (1/2, 1)$, $\varrho_0 \in W_q^2(\Omega)$, $3 < q \leq \frac{3}{3/2-s}$. Assume also that there exist positive constants $\varrho_* < \varrho^*$ such that $\varrho_* \leq \varrho_0 \leq \varrho^*$. Then the mapping (5.2) has a fixed point belonging to $H^{2+s, 1+s/2}(\Omega^T)$.*

Remark 5.2. In Lemma 4.4 there is assumption that $\partial_t^{s/2} \varrho_x|_{t=0} \in L_r(\Omega)$, where $r \leq 6$ and is such that $1 \leq s + \frac{3}{r}$ for $s \in (1/2, 1)$, $r \leq q$.

We see that the condition is satisfied in view of the relations

$$\varrho_0 \in W_q^2(\Omega), \quad r \leq q, \quad v_0 \in H^{1+s}(\Omega).$$

We have

$$\begin{aligned} \|\partial_t^{s/2} \varrho_x|_{t=0}\|_{L_r(\Omega)} &\leq c(\|\partial_t \varrho_x|_{t=0}\|_{L_r(\Omega)} + \|\varrho_x|_{t=0}\|_{L_r(\Omega)}) \\ &\leq c\|(v_x \cdot \nabla \varrho + v \cdot \nabla \varrho_x)|_{t=0}\|_{L_r(\Omega)} + c\|\varrho_{0,x}\|_{L_r(\Omega)} \\ &\leq c(\|v_0\|_{H^{1+s}(\Omega)} \|\varrho_{0,x}\|_{W_q^1(\Omega)} + \|\varrho_{0,x}\|_{L_q(\Omega)}). \end{aligned}$$

Proof of Lemma 5.1. In view of the a priori estimate (4.69) we have to examine the other assumptions of the Leray-Schauder fixed point theorem.

For $\lambda = 0$ we have the existence of a unique solution.

We have that

$$\varrho_* \leq \varrho_0(1 - \lambda) + \lambda \varrho(\tilde{v}) \leq \varrho^*$$

since ϱ_0 is continuous and $\varrho_0 \in W_p^1(\Omega)$, $p > 3$ we have that $\varrho = \varrho(\tilde{v})$ is continuous because

$$\begin{aligned} \|\varrho_x(t)\|_{L_p(\Omega)} &\leq \exp(\|\tilde{v}\|_{L_1(0,t;L_\infty(\Omega))}) \|\varrho_x(0)\|_{L_p(\Omega)}, \\ (5.4) \quad \|\varrho_t(t)\|_{L_p(\Omega)} &\leq \|\tilde{v}\|_{L_\infty(\Omega^t)} \sup_t \|\varrho_x\|_{L_p(\Omega)} \\ &\leq \|\tilde{v}\|_{L_\infty(\Omega^t)} \exp(\|\tilde{v}\|_{L_1(0,t;L_\infty(\Omega))}) \|\varrho_x(0)\|_{L_p(\Omega)}. \end{aligned}$$

Then

$$\begin{aligned} (5.5) \quad \sup_{x,x',t} \frac{|\varrho(x,t) - \varrho(x',t)|}{|x - x'|^{1/p'}} &\leq \sup \frac{|\varrho(x_1, x_2, x_3, t) - \varrho(x'_1, x_2, x_3, t)|}{|x_1 - x'_1|^{1/p'}} \\ &\quad + \sup \frac{|\varrho(x'_1, x_2, x_3, t) - \varrho(x'_1, x'_2, x_3, t)|}{|x_2 - x'_2|^{1/p'}} \\ &\quad + \sup \frac{|\varrho(x'_1, x'_2, x_3, t) - \varrho(x'_1, x'_2, x'_3, t)|}{|x_3 - x'_3|^{1/p'}} \\ &\leq \sup \left(\int_{x'_1}^{x_1} |\varrho_z(z, x_2, x_3, t)|^p dz \right)^{1/p} + \sup \left(\int_{x'_2}^{x_2} |\varrho_z(x_1, z, x_3, t)|^p dz \right)^{1/p} \\ &\quad + \sup \left(\int_{x'_3}^{x_3} |\varrho_z(x_1, x_2, z, t)|^p dz \right)^{1/p} \leq c \sup_t \|\varrho_x\|_{L_\infty(\Omega)}, \end{aligned}$$

where $x'_i < x_i$, $i = 1, 2, 3$. Moreover,

$$(5.6) \quad \frac{|\varrho(x, t) - \varrho(x, t')|}{|t - t'|^{1/p'}} \leq \left(\int_{t'}^t |\varrho_t|^p dt \right)^{1/p}.$$

Hence we have continuity of ϱ if $\varrho_x(0) \in L_\infty(\Omega)$.

Assuming that $\tilde{v} \in W_2^{2+\sigma, 1+\sigma/2}(\Omega^T)$, $\sigma \in (1/2, 1)$ we show that

$$\|\varrho \tilde{v} \cdot \nabla \tilde{v}\|_{W_2^{s, s/2}(\Omega^T)} \leq \varphi(\|\tilde{v}\|_{W_2^{2+\sigma, 1+\sigma/2}(\Omega^T)}),$$

where $1 > s > \sigma$.

We have to assume that $\sigma > 1/2$ because we need the imbeddings

$$\|\tilde{v}\|_{L_\infty(\Omega^T)} + \|\tilde{v}_x\|_{L_2(0,T;L_\infty(\Omega))} \leq c\|\tilde{v}\|_{W_2^{2+\sigma,1+\sigma/2}(\Omega^T)}.$$

Hence, we have shown that

$$(5.7) \quad \Phi : W_2^{2+\sigma,1+\sigma/2}(\Omega^T) \times [0,1] \rightarrow W_2^{2+s,1+s/2}(\Omega^T) \subset W_2^{2+\sigma,1+\sigma/2}(\Omega^T)$$

where the last imbedding is compact. Then mapping Φ is compact.

To show continuity of mapping Φ we introduce the differences

$$(5.8) \quad \tilde{V} = \tilde{v}_1 - \tilde{v}_2, \quad V = v_1 - v_2, \quad P = p_1 - p_2.$$

Then (V, P) is a solution to the problem

$$(5.9) \quad \begin{aligned} & [\varrho_0(1-\lambda) + \lambda\varrho(\tilde{v}_2)]V_t - \operatorname{div} \mathbb{T}(V, P) \\ &= -\lambda(\varrho(\tilde{v}_1) - \varrho(\tilde{v}_2))v_{1t} - (\varrho(\tilde{v}_1)\tilde{v}_1 \cdot \nabla \tilde{v}_1 - \varrho(\tilde{v}_2)\tilde{v}_2 \cdot \nabla \tilde{v}_2) \\ & \quad + (\varrho(\tilde{v}_1) - \varrho(\tilde{v}_2))f, \\ & \operatorname{div} V = 0, \\ & V \cdot \bar{n}|_S = 0, \quad \bar{n} \cdot \mathbb{D}(V) \cdot \bar{\tau}_\alpha + \delta_{1j}\gamma V \cdot \bar{\tau}_\alpha|_{S_j} = 0, \quad \alpha = 1, 2, \quad j = 1, 2, \\ & V|_{t=0} = 0. \end{aligned}$$

From (5.9) we have

$$(5.10) \quad \|V\|_{W_2^{2+s,1+s/2}(\Omega^T)} \leq \varphi(A)\|\tilde{V}\|_{W_2^{2+\sigma,1+\sigma/2}(\Omega^T)},$$

where

$$\sum_{i=1}^2 \|\tilde{v}_i\|_{W_2^{2+s,1+s/2}(\Omega^T)} \leq A,$$

where the last estimate is shown in Section 4.

Let us examine continuity of Φ with respect to λ . For this purpose we examine

$$(5.11) \quad \begin{aligned} & [\varrho_0(1-\lambda_i) + \lambda_i\varrho(\tilde{v})]v_{it} - \operatorname{div} \mathbb{T}(v_i, p_i) \\ &= \lambda_i[-\varrho(\tilde{v})\tilde{v} \cdot \nabla \tilde{v} + \varrho(\tilde{v})f], \\ & \operatorname{div} v_i = 0 \\ & \bar{n} \cdot v_i|_S = 0, \quad \bar{n} \cdot \mathbb{D}(v_i) \cdot \bar{\tau}_\alpha + \delta_{1j}\gamma v_i \cdot \bar{\tau}_\alpha|_{S_j} = 0, \quad \alpha = 1, 2, \quad j = 1, 2, \\ & v_i|_{t=0} = v_0, \end{aligned}$$

where $i = 1, 2$.

Introducing the differences

$$(5.12) \quad V = v_1 - v_2, \quad P = p_1 - p_2, \quad \Lambda = \lambda_1 - \lambda_2$$

we see that they satisfy the problem

$$(5.13) \quad \begin{aligned} & [\varrho_0(1 - \lambda_2) + \lambda_2 \varrho(\tilde{v})] V_t - \operatorname{div} \mathbb{T}(V, P) \\ & = \Lambda(\varrho_0 - \varrho(\tilde{v})) v_{1t} + \Lambda(-\varrho(\tilde{v}) \tilde{v} \cdot \nabla \tilde{v} + \varrho(\tilde{v}) f), \\ & \operatorname{div} V = 0 \\ & V \cdot \bar{n}|_S = 0, \quad \nu \bar{n} \cdot \mathbb{D}(V) \cdot \bar{\tau}_\alpha + \delta_{1j} \gamma V \cdot \bar{\tau}_\alpha|_{S_j} = 0, \quad \alpha = 1, 2, \quad j = 1, 2, \\ & V|_{t=0} = 0. \end{aligned}$$

For solutions to (5.13) we have

$$(5.14) \quad \begin{aligned} & \|V\|_{W_2^{2+s, 1+s/2}(\Omega^T)} + \|\nabla P\|_{W_2^{s, s/2}(\Omega^T)} \\ & \leq \varphi(A)(1 + \|v_{1t}\|_{W_2^{s, s/2}(\Omega^T)}) \Lambda. \end{aligned}$$

Hence, the continuity with respect to λ follows.

Applying the Leray-Schauder fixed point theorem we prove Lemma 5.1.

This concludes the proof.

Now we prove uniqueness

Lemma 5.3. *Assume that $\varrho_* \leq \varrho \leq \varrho^*$, $\varrho \in L_2(0, T; H^2(\Omega))$, $v \in H^{2,1}(\Omega^T)$, $f \in L_2(\Omega^T)$. Then we have uniqueness of solutions to problem (1.1).*

Proof. Assume that we have two solutions (v_i, ϱ_i, p_i) , $i = 1, 2$, to problem (1.1). Let

$$(5.15) \quad V = v_1 - v_2, \quad R = \varrho_1 - \varrho_2, \quad P = p_1 - p_2.$$

Then functions (5.15) are solutions to the problem

$$(5.16) \quad \begin{aligned} & \varrho_1 V_t + \varrho_1 v_1 \cdot \nabla V + \varrho_1 V \cdot \nabla v_2 - \operatorname{div} \mathbb{T}(V, P) \\ & = Rf - R(v_{2t} + v_2 \cdot \nabla v_2), \\ & \operatorname{div} V = 0, \\ & \varrho_1 R_t + \varrho_1 v_1 \cdot \nabla R + \varrho_1 V \cdot \nabla \varrho_2 = 0, \\ & V \cdot \bar{n}|_S = 0, \quad (\nu \bar{n} \cdot \mathbb{T}(V, P) \cdot \bar{\tau}_\alpha + \delta_{1j} \gamma V \cdot \bar{\tau}_\alpha)|_{S_j} = 0, \quad \alpha = 1, 2, \quad j = 1, 2, \\ & V|_{t=0} = 0, \quad R|_{t=0} = 0. \end{aligned}$$

Multiplying (5.16)₁ by V , integrating over Ω , and employing the equation of continuity for ϱ_1 and using the Korn inequality we obtain

$$(5.17) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho_1 V^2 dx + \|V\|_{H^1(\Omega)}^2 + \gamma \int_{S_1} |V \cdot \bar{\tau}_\alpha|^2 dS_1 \\ & \leq - \int_{\Omega} V \cdot \nabla v_2 \cdot V dx + \int_{\Omega} Rf \cdot V dx - \int_{\Omega} R(v_{2t} + v_2 \cdot \nabla v_2) \cdot V dx. \end{aligned}$$

The first term on the r.h.s. of (5.17) is estimated by

$$\varepsilon \|V\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \|\nabla v_2\|_{L_3(\Omega)}^2 \|V\|_{L_2(\Omega)}^2,$$

the second by

$$\varepsilon \|V\|_{L_6(\Omega)}^2 + c(1/\varepsilon) \|f\|_{L_2(\Omega)}^2 \|R\|_{L_3(\Omega)}^2$$

and the last by

$$\varepsilon \|V\|_{L_6(\Omega)}^2 + c(1/\varepsilon) (\|v_{2t}\|_{L_2(\Omega)}^2 + \|v_2\|_{L_4(\Omega)}^2 \|\nabla v_2\|_{L_4(\Omega)}^2) \|R\|_{L_3(\Omega)}^2.$$

Using the estimates in (5.17) and assuming that ε is sufficiently small we arrive to the inequality

$$(5.18) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} \varrho_1 V^2 dx + \|V\|_{H^1(\Omega)}^2 \leq c \|\nabla v_2\|_{L_3(\Omega)}^2 \|V\|_{L_2(\Omega)}^2 \\ & + c(\|f\|_{L_2(\Omega)}^2 + \|v_{2t}\|_{L_2(\Omega)}^2 + \|v_2\|_{L_4(\Omega)}^2 \|\nabla v_2\|_{L_4(\Omega)}^2) \|R\|_{L_3(\Omega)}^2. \end{aligned}$$

Multiplying (5.16)₃ by R^2 , integrating over Ω and using the equation of continuity for ϱ_1 yields

$$(5.19) \quad \frac{1}{3} \frac{d}{dt} \int_{\Omega} \varrho_1 R^3 dx = - \int_{\Omega} \varrho_1 V \cdot \nabla \varrho_2 R^2 dx.$$

Estimating the r.h.s. by

$$\begin{aligned} & \left(\int_{\Omega} |V \cdot \nabla \varrho_2|^3 dx \right)^{1/3} \left(\int_{\Omega} (\varrho_1 R^2)^{3/2} dx \right)^{2/3} \\ & \leq \varphi(\varrho^*) \|V\|_{L_6(\Omega)} \|\nabla \varrho_2\|_{L_6(\Omega)} \left(\int_{\Omega} \varrho_1 R^3 dx \right)^{2/3} \end{aligned}$$

we obtain from (5.19) the inequality

$$\begin{aligned}
(5.20) \quad & \frac{d}{dt} \left(\int_{\Omega} \varrho_1 R^3 dx \right)^{2/3} \leq \varphi(\varrho^*) \|V\|_{L_6(\Omega)} \|\nabla \varrho_2\|_{L_6(\Omega)} \left(\int_{\Omega} \varrho_1 R^3 dx \right)^{1/3} \\
& \leq \varepsilon \|V\|_{L_6(\Omega)}^2 + \varphi(1/\varepsilon, \varrho^*) \|\nabla \varrho_2\|_{L_6(\Omega)}^2 \left(\int_{\Omega} \varrho_1 R^3 dx \right)^{2/3}
\end{aligned}$$

Adding (5.18) and (5.20) with ε sufficiently small and defining the quantities

$$\begin{aligned}
Y(t) &= \int_{\Omega} \varrho_1 V^2 dx + \left(\int_{\Omega} \varrho_1 R^3 dx \right)^{2/3}, \\
A(t) &= \|\nabla \varrho_2\|_{L_6(\Omega)}^2 + \|v_{2t}\|_{L_2(\Omega)}^2 + \|v_2\|_{L_4(\Omega)}^2 \|\nabla v_2\|_{L_4(\Omega)}^2 + \|f\|_{L_2(\Omega)}^2
\end{aligned}$$

we obtain

$$(5.21) \quad \frac{d}{dt} Y \leq \varphi(\varrho^*, \varrho_*) AY$$

Hence for $I = \int_0^T A dt < \infty$ we have uniqueness.

Moreover, we see that

$$\begin{aligned}
I &\leq \|\varrho_2\|_{L_2(0,T;H^2(\Omega))}^2 + \|v_2\|_{H^{2,1}(\Omega^T)}^2 (1 + \|v_2\|_{H^{2,1}(\Omega^T)}^2) \\
&\quad + \|f\|_{L_2(\Omega^T)}^2 < \infty.
\end{aligned}$$

This concludes the proof.

Remark 5.4. The assumptions of Lemma 5.3 are satisfied in view of the assumptions of Lemma 5.1.

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